Asymptotics of Laurent Polynomials of Even Degree Orthogonal with Respect to Varying Exponential Weights

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Abstract

Let $\Lambda^{\mathbb{R}}$ denote the linear space over $\mathbb{R}$ spanned by z^k, $k \in \mathbb{Z}$. Define the real inner product (with varying exponential weights) $\langle \cdot, \cdot \rangle_{\mathcal{L}} : \Lambda^{\mathbb{R}} \times \Lambda^{\mathbb{R}} \to \mathbb{R}$, $(f, g) \mapsto \int_{\mathbb{R}} f(s)g(s) \exp(-\mathcal{N}\, V(s))\, \mathrm{d}s$, $\mathcal{N} \in \mathbb{N}$, where the external field V satisfies: (i) V is real analytic on $\mathbb{R} \setminus \{0\}$; (ii) $\lim_{|x| \to \infty}(V(x)/\ln(x^2+1)) = +\infty$; and (iii) $\lim_{|x| \to 0}(V(x)/\ln(x^{-2}+1)) = +\infty$. Orthogonalisation of the (ordered) base $\{1, z^{-1}, z, z^{-2}, z^2, \dots, z^{-k}, z^k, \dots\}$ with respect to $\langle \cdot, \cdot \rangle_{\mathcal{L}}$ yields the even degree and odd degree orthonormal Laurent polynomials $\{\phi_m(z)\}_{m=0}^{\infty}$: $\phi_{2n}(z) = \xi^{(2n)}_{-n} z^{-n} + \cdots + \xi^{(2n)}_{n} z^n$, $\xi^{(2n)}_{n} > 0$, and $\phi_{2n+1}(z) = \xi^{(2n+1)}_{-n-1} z^{-n-1} + \cdots + \xi^{(2n+1)}_{n} z^n$, $\xi^{(2n+1)}_{-n-1} > 0$. Define the even degree and odd degree monic orthogonal Laurent polynomials: $\boldsymbol{\pi}_{2n}(z) := (\xi^{(2n)}_{n})^{-1} \phi_{2n}(z)$ and $\boldsymbol{\pi}_{2n+1}(z) := (\xi^{(2n+1)}_{-n-1})^{-1} \phi_{2n+1}(z)$. Asymptotics in the double-scaling limit as $\mathcal{N}, n \to \infty$ such that $\mathcal{N}/n = 1 + o(1)$ of $\boldsymbol{\pi}_{2n}(z)$ (in the entire complex plane), $\xi^{(2n)}_{n}$, $\phi_{2n}(z)$ (in the entire complex plane), and Hankel determinant ratios associated with the real-valued, bi-infinite, strong moment sequence $\left\{c_k = \int_{\mathbb{R}} s^k \exp(-\mathcal{N}\, V(s))\, \mathrm{d}s\right\}_{k \in \mathbb{Z}}$ are obtained by formulating the even degree monic orthogonal Laurent polynomial problem as a matrix Riemann-Hilbert problem on $\mathbb{R}$, and then extracting the large-n behaviour by applying the non-linear steepest-descent method introduced in [1] and further developed in [2, 3].

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1 Introduction and Background

Consider the *classical Stieltjes* (resp., *classical Hamburger*) *moment problem* (SMP) (resp., HMP): given a simply-infinite (moment) sequence of real numbers $\{c_n\}_{n=0}^{\infty}$:

(i) find necessary and sufficient conditions for the existence of a non-negative Borel measure $\mu_{\mathrm{MP}}^{\mathrm{S}}$ (resp., $\mu_{\mathrm{MP}}^{\mathrm{H}}$) on $[0,+\infty)$ (resp., $(-\infty,+\infty)$), and with infinite support, such that $c_n = \int_0^{+\infty} t^n \,\mathrm{d}\mu_{\mathrm{MP}}^{\mathrm{S}}(t)$, $n \in \mathbb{Z}_0^+ := \{0\} \cup \mathbb{N}$ (resp., $c_n = \int_{-\infty}^{+\infty} t^n \,\mathrm{d}\mu_{\mathrm{MP}}^{\mathrm{H}}(t)$, $n \in \mathbb{Z}_0^+$), where the (improper) integral is to be understood in the Riemann-Stieltjes sense;

(ii) when there is a solution of the existence problem, in which case the SMP (resp., HMP) is *determinate*, find conditions for the uniqueness of the solution; and

(iii) when there is more than one solution, in which case the SMP (resp., HMP) is *indeterminate*, describe the family of all solutions.

The SMP was—first—treated in 1894/95 by Stieltjes in the pioneering works [4], and the HMP was introduced and solved in 1920/21 by Hamburger in the landmark works [5]. The subsequent development of the theory of moment problems brought forth the profound fact that, over and above the indispensable utility afforded by the analytic theory of continued fractions, in particular, *S*- and real *J*-fractions, the theory of orthogonal polynomials [6] played a seminal, intimate and central rôle (see, for example, [7]).

Questions regarding two simply-infinite (moment) sequences $\{c_n\}_{n\in\mathbb{Z}_0^+}$ and $\{c_{-n}\}_{n\in\mathbb{N}}$ of real numbers, or, equivalently, doubly- or bi-infinite (moment) sequences $\{c_n\}_{n\in\mathbb{Z}}$ of real numbers, manifest, in various settings, purely mathematical and/or otherwise, as natural extensions of the foregoing. This generalisation is colloquially refered to as the *strong Stieltjes* (resp., *strong Hamburger*) *moment problem* (SSMP) (resp., SHMP), namely, given a doubly- or bi-infinite (moment) sequence $\{c_n\}_{n\in\mathbb{Z}}$ of real numbers:

(1) find necessary and sufficient conditions for the existence of a non-negative measure $\mu_{\mathrm{MP}}^{\mathrm{SS}}$ (resp., $\mu_{\mathrm{MP}}^{\mathrm{SH}}$) on $[0,+\infty)$ (resp., $(-\infty,+\infty)$), and with infinite support, such that $c_n = \int_0^{+\infty} t^n \,\mathrm{d}\mu_{\mathrm{MP}}^{\mathrm{SS}}(t)$, $n\in\mathbb{Z}$ (resp., $c_n = \int_{-\infty}^{+\infty} t^n \,\mathrm{d}\mu_{\mathrm{MP}}^{\mathrm{SH}}(t)$, $n\in\mathbb{Z}$), where the (improper) integral is to be understood in the sense of Riemann-Stieltjes;

(2) when there is a solution, in which case the SSMP (resp., SHMP) is determinate, find conditions for the uniqueness of the solution; and

(3) when there is more than one solution, in which case the SSMP (resp., SHMP) is indeterminate, describe the family of all solutions.

The SSMP (resp., SHMP) was introduced in 1980 (resp., 1981) by Jones *et al.* [8] (resp., Jones *et al.* [9]), and studied further in [10–14] (see, also, the review article [15]). Unlike the moment theory for the SMP and the HMP, wherein the theory of orthogonal polynomials, and the analytic theory of continued fractions, enjoyed a prominent rôle, the extension of the moment theory to the SSMP and the SHMP introduced a 'rational generalisation' of the orthogonal polynomials, namely, the *orthogonal Laurent* (or *L*-) *polynomials* (as well as the introduction of special kinds of continued fractions commonly referred to as positive-*T* fractions), which are discussed below [10–21]. (The SHMP can also be solved using the spectral theory of unbounded self-adjoint operators in Hilbert space [22]; see, also, [23].)

For any pair $(p,q)\in\mathbb{Z}\times\mathbb{Z}$, with $p\leqslant q$, let $\Lambda_{p,q}^{\mathbb{C}} := \left\{f\colon \mathbb{C}^* \to \mathbb{C};\ f(z)=\sum_{k=p}^{q}\widehat{\lambda}_k z^k,\ \widehat{\lambda}_k\in\mathbb{C},\ k=p,\dots,q\right\}$, where $\mathbb{C}^* := \mathbb{C}\setminus\{0\}$. For any $m\in\mathbb{Z}_0^+$, set $\Lambda_{2m}^{\mathbb{C}} := \Lambda_{-m,m}^{\mathbb{C}}$, $\Lambda_{2m+1}^{\mathbb{C}} := \Lambda_{-m-1,m}^{\mathbb{C}}$, and $\Lambda^{\mathbb{C}} := \cup_{m\in\mathbb{Z}_0^+}(\Lambda_{2m}^{\mathbb{C}} \cup \Lambda_{2m+1}^{\mathbb{C}})$. A function (or element) $f\in\Lambda^{\mathbb{C}}$ is called a *Laurent* (or *L*-) *polynomial*. (Note: the sets $\Lambda_{p,q}^{\mathbb{C}}$ and $\Lambda^{\mathbb{C}}$ form linear spaces over the field $\mathbb{C}$ with respect to the operations of addition and multiplication by a scalar.) Bases for each of the spaces $\Lambda_{2m}^{\mathbb{C}}$, $\Lambda_{2m+1}^{\mathbb{C}}$, and $\Lambda^{\mathbb{C}}$, respectively, are $\{z^{-m},\dots,z^m\}$, $\{z^{-m-1},\dots,z^m\}$, and $\{\text{const.}, z^{-1}, z, z^{-2}, z^2, \dots, z^{-k}, z^k, \dots\}$ (the basis for $\Lambda^{\mathbb{C}}$ corresponds to the *cyclically-repeated pole sequence* $\{\text{no pole}, 0, \infty, 0, \infty, \dots, 0, \infty, \dots\}$). Furthermore, note that, for each $0 \not\equiv f \in \Lambda^{\mathbb{C}}$, there exists a unique $l\in\mathbb{Z}_0^+$ such that $f\in\Lambda_l^{\mathbb{C}}$. For $l\in\mathbb{Z}_0^+$ and $0\not\equiv f\in\Lambda_l^{\mathbb{C}}$, the *L-degree* of f, symbolically $LD(f)$, is defined as

$$LD(f) := l.$$

For $\Lambda^{\mathbb{C}} \ni f = \sum_{j\in\mathbb{Z}} \widehat{\lambda}_j z^j$, set $C_j(f) := \widehat{\lambda}_j$, $j\in\mathbb{Z}$. For each $l\in\mathbb{Z}_0^+$ and $0 \not\equiv f \in \Lambda_l^{\mathbb{C}}$, define the *leading coefficient* of f, symbolically $LC(f)$, and the *trailing coefficient* of f, symbolically $TC(f)$, as follows:

$$LC(f) := \begin{cases} \widehat{\lambda}_m, & l=2m, \\ \widehat{\lambda}_{-m-1}, & l=2m+1, \end{cases}$$

and

$$TC(f) := \begin{cases} \widehat{\lambda}_{-m}, & l=2m, \\ \widehat{\lambda}_m, & l=2m+1. \end{cases}$$

Thus, for $l\in\mathbb{Z}_0^+$ and $0 \not\equiv f \in \Lambda_l^{\mathbb{C}}$, one writes, for $f := f_l(z)$: (1) if $l=2m$,

$$f_{2m}(z) = TC(f)z^{-m} + \cdots + LC(f)z^m;$$

and (2) if $l=2m+1$,

$$f_{2m+1}(z) = LC(f)z^{-m-1} + \cdots + TC(f)z^m.$$

For $l\in\mathbb{Z}_0^+$, $0 \not\equiv f \in \Lambda_l^{\mathbb{C}}$ is called *monic* if $LC(f)=1$.

Consider the positive measure on $\mathbb{R}$ (oriented throughout this work, unless stated otherwise, from $-\infty$ to $+\infty$) given by

$$\mathrm{d}\widetilde{\mu}(z) = \widetilde{w}(z)\mathrm{d}z,$$

with varying exponential weight function of the form

$$\widetilde{w}(z) = \exp(-\mathcal{N}\, V(z)), \quad \mathcal{N}\in\mathbb{N},$$

where the *external field* $V \colon \mathbb{R}\setminus\{0\} \to \mathbb{R}$ satisfies the following conditions:

$$V \text{ is real analytic on } \mathbb{R}\setminus\{0\}; \tag{V1}$$

$$\lim_{|x|\to\infty}\Big(V(x)/\ln(x^2+1)\Big) = +\infty; \tag{V2}$$

$$\lim_{|x|\to 0}\Big(V(x)/\ln(x^{-2}+1)\Big) = +\infty. \tag{V3}$$

(For example, a rational function of the form $V(z) = \sum_{k=-2m_1}^{2m_2} \varrho_k z^k$, with $\varrho_k \in \mathbb{R}$, $k = -2m_1, \ldots, 2m_2$, $m_{1,2} \in \mathbb{N}$, and $\varrho_{-2m_1}, \varrho_{2m_2} > 0$ would suffice.) Define (uniquely) the *strong moment linear functional* $\mathcal{L}$ by its action on the basis elements of $\Lambda^{\mathbb{C}}$: $\mathcal{L} \colon \Lambda^{\mathbb{C}} \to \Lambda^{\mathbb{C}}$, $f = \sum_{k\in\mathbb{Z}} \widehat{\lambda}_k z^k \mapsto \mathcal{L}(f) := \sum_{k\in\mathbb{Z}} \widehat{\lambda}_k c_k$, where $c_k = \mathcal{L}(z^k) = \int_{\mathbb{R}} s^k \exp(-\mathcal{N}\, V(s))\,\mathrm{d}s$, $(k,\mathcal{N}) \in \mathbb{Z}\times\mathbb{N}$. (Note that, as per the discussion above, $\Big\{c_k = \int_{\mathbb{R}} s^k \exp(-\mathcal{N}\, V(s))\,\mathrm{d}s,\ \mathcal{N}\in\mathbb{N}\Big\}_{k\in\mathbb{Z}}$ is a bi-infinite, real-valued, *strong moment sequence*: c_k is called the *kth strong moment of* $\mathcal{L}$.) Associated with the above-defined bi-infinite, real-valued, strong moment sequence $\{c_k\}_{k\in\mathbb{Z}}$ are the *Hankel determinants* $H_k^{(m)}$, $(m,k)\in\mathbb{Z}\times\mathbb{N}$ [10, 11, 15, 17]:

$$H_0^{(m)} := 1 \qquad \text{and} \qquad H_k^{(m)} := \begin{vmatrix} c_m & c_{m+1} & \cdots & c_{m+k-2} & c_{m+k-1} \\ c_{m+1} & c_{m+2} & \cdots & c_{m+k-1} & c_{m+k} \\ c_{m+2} & c_{m+3} & \cdots & c_{m+k} & c_{m+k+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{m+k-1} & c_{m+k} & \cdots & c_{m+2k-3} & c_{m+2k-2} \end{vmatrix}. \tag{1.1}$$

For any pair $(p,q)\in\mathbb{Z}\times\mathbb{Z}$, with $p\leqslant q$, let $\Lambda_{p,q}^{\mathbb{R}} := \Big\{f \colon \mathbb{C}^* \to \mathbb{C};\ f(z) = \sum_{k=p}^{q} \widetilde{\lambda}_k z^k,\ \widetilde{\lambda}_k \in \mathbb{R},\ k=p,\ldots,q\Big\}$, and define, analogously as above, for $m\in\mathbb{Z}_0^+$, $\Lambda_{2m}^{\mathbb{R}} := \Lambda_{-m,m}^{\mathbb{R}}$, $\Lambda_{2m+1}^{\mathbb{R}} := \Lambda_{-m-1,m}^{\mathbb{R}}$, and $\Lambda^{\mathbb{R}} := \cup_{m\in\mathbb{Z}_0^+}(\Lambda_{2m}^{\mathbb{R}} \cup \Lambda_{2m+1}^{\mathbb{R}})$. (Note: the sets $\Lambda_{p,q}^{\mathbb{R}}$ and $\Lambda^{\mathbb{R}}$ form linear spaces over the field $\mathbb{R}$ with respect to the operations of addition and multiplication by a scalar; furthermore, $\Lambda^{\mathbb{R}}$ ($\subset \Lambda^{\mathbb{C}}$) is the linear space over $\mathbb{R}$ spanned by z^j, $j\in\mathbb{Z}$.) Hereafter, we shall be concerned only with (real) L-polynomials in $\Lambda^{\mathbb{R}}$: the—ordered—base for $\Lambda^{\mathbb{R}}$ is $\{1, z^{-1}, z, z^{-2}, z^2, \ldots, z^{-k}, z^k, \ldots\}$, corresponding to the cyclically-repeated pole sequence $\{\text{no pole}, 0, \infty, 0, \infty, \ldots, 0, \infty, \ldots\}$. Define the real bilinear form $\langle\cdot,\cdot\rangle_{\mathcal{L}}$ as follows: $\langle\cdot,\cdot\rangle_{\mathcal{L}} \colon \Lambda^{\mathbb{R}}\times\Lambda^{\mathbb{R}} \to \mathbb{R}$, $(f,g) \mapsto \langle f,g\rangle_{\mathcal{L}} := \mathcal{L}(f(z)g(z)) = \int_{\mathbb{R}} f(s)g(s)\mathrm{e}^{-\mathcal{N}\, V(s)}\,\mathrm{d}s$, $\mathcal{N}\in\mathbb{N}$. It is a fact [10, 11, 15, 17] that the bilinear form $\langle\cdot,\cdot\rangle_{\mathcal{L}}$ thus defined is an inner product if and only if $H_{2m}^{(-2m)} > 0$ and $H_{2m+1}^{(-2m)} > 0$ $\forall\ m\in\mathbb{Z}_0^+$ (see Equations (1.8) below, and Subsection 2.2, the proof of Lemma 2.2.1); and this fact is used, with little or no further reference, throughout this work (see, also, [24]).

Remark 1.1. These latter two (Hankel determinant) inequalities also appear when the question of the solvability of the SHMP is posed (in this case, the c_k, $k \in \mathbb{Z}$, which appear in Equations (1.1) should be replaced by c_k^{SHMP}, $k \in \mathbb{Z}$): indeed, if these two inequalities are true $\forall\ m \in \mathbb{Z}_0^+$, then there is a non-negative measure $\mu_{\mathrm{MP}}^{\mathrm{SH}}$ (on $\mathbb{R}$) with the given (real) moments. For the case of the SSMP, there are four (Hankel determinant) inequalities (in this latter case, the c_k, $k \in \mathbb{Z}$, which appear in Equations (1.1) should be replaced by c_k^{SSMP}, $k \in \mathbb{Z}$) which guarantee the existence of a non-negative measure $\mu_{\mathrm{MP}}^{\mathrm{SS}}$ (on $[0, +\infty)$) with the given moments, namely [8] (see, also, [10, 11]): for each $m \in \mathbb{Z}_0^+$, $H_{2m}^{(-2m)} > 0$, $H_{2m+1}^{(-2m)} > 0$, $H_{2m}^{(-2m+1)} > 0$, and $H_{2m+1}^{(-2m-1)} < 0$. It is interesting to note that the former solvability conditions do not automatically imply that the positive (real) moments $\{c_k^{\mathrm{SHMP}}\}_{k \in \mathbb{Z}_0^+}$ determine a measure via the HMP: a similar statement holds true for the SMP (see the latter four solvability conditions). ■

If $f \in \Lambda^{\mathbb{R}}$, then

$$\|f(\cdot)\|_{\mathcal{L}} := (\langle f, f \rangle_{\mathcal{L}})^{1/2}$$

is called the *norm of f with respect to* $\mathcal{L}$: note that $\|f(\cdot)\|_{\mathcal{L}} \geqslant 0\ \forall\ f \in \Lambda^{\mathbb{R}}$, and $\|f(\cdot)\|_{\mathcal{L}} > 0$ if $0 \not\equiv f \in \Lambda^{\mathbb{R}}$. $\{\phi_n^{\flat}(z)\}_{n \in \mathbb{Z}_0^+}$ is called a (real) orthonormal Laurent (or L-) polynomial sequence (ONLPS) with respect to $\mathcal{L}$ if, $\forall\ m, n \in \mathbb{Z}_0^+$:

(i) $\phi_n^{\flat} \in \Lambda_n^{\mathbb{R}}$, that is, $LD(\phi_n^{\flat}) := n$;

(ii) $\langle \phi_m^{\flat}, \phi_{n'}^{\flat} \rangle_{\mathcal{L}} = 0\ \forall\ m \neq n'$, or, alternatively, $\langle f, \phi_n^{\flat} \rangle_{\mathcal{L}} = 0\ \forall\ f \in \Lambda_{n-1}^{\mathbb{R}}$;

(iii) $\langle \phi_m^{\flat}, \phi_m^{\flat} \rangle_{\mathcal{L}} =: \|\phi_m^{\flat}(\cdot)\|_{\mathcal{L}}^2 = 1$.

Orthonormalisation of $\{1, z^{-1}, z, z^{-2}, z^2, \ldots, z^{-n}, z^n, \ldots\}$, corresponding to the cyclically-repeated pole sequence $\{\text{no pole}, 0, \infty, 0, \infty, \ldots, 0, \infty, \ldots\}$, with respect to $\langle \cdot, \cdot \rangle_{\mathcal{L}}$ via the Gram-Schmidt orthogonalisation method, leads to the ONLPS, or, simply, orthonormal Laurent (or L-) polynomials (OLPs), $\{\phi_m(z)\}_{m \in \mathbb{Z}_0^+}$, which, by suitable normalisation, may be written as, for $m = 2n$,

$$\phi_{2n}(z) = \xi_{-n}^{(2n)} z^{-n} + \cdots + \xi_n^{(2n)} z^n, \qquad \xi_n^{(2n)} > 0, \tag{1.2}$$

and, for $m = 2n+1$,

$$\phi_{2n+1}(z) = \xi_{-n-1}^{(2n+1)} z^{-n-1} + \cdots + \xi_n^{(2n+1)} z^n, \qquad \xi_{-n-1}^{(2n+1)} > 0. \tag{1.3}$$

The ϕ_n's are normalised so that they all have real coefficients; in particular, the leading coefficients, $LC(\phi_{2n}) := \xi_n^{(2n)}$ and $LC(\phi_{2n+1}) := \xi_{-n-1}^{(2n+1)}$, $n \in \mathbb{Z}_0^+$, are both positive, $\xi_0^{(0)} = 1$, and $\phi_0(z) \equiv 1$. Even though the leading coefficients, $\xi_n^{(2n)}$ and $\xi_{-n-1}^{(2n+1)}$, $n \in \mathbb{Z}_0^+$, are non-zero (in particular, they are positive), no such restriction applies to the trailing coefficients, $TC(\phi_{2n}) := \xi_{-n}^{(2n)}$ and $TC(\phi_{2n+1}) := \xi_n^{(2n+1)}$, $n \in \mathbb{Z}_0^+$. Furthermore, note that, by construction:

(1) $\langle \phi_{2n}, z^j \rangle_{\mathcal{L}} = 0$, $j = -n, \ldots, n-1$;

(2) $\langle \phi_{2n+1}, z^j \rangle_{\mathcal{L}} = 0$, $j = -n, \ldots, n$;

(3) $\langle \phi_j, \phi_k \rangle_{\mathcal{L}} = \delta_{jk}$, $j, k \in \mathbb{Z}_0^+$, where δ_{jk} is the Kronecker delta.

Moreover, if, for each $m \in \mathbb{Z}_0^+$, the orthonormal L-polynomials $\phi_{2m}(z)$ and $\phi_{2m+1}(z)$, respectively, are such that $TC(\phi_{2m}) := \xi_{-m}^{(2m)} \neq 0$ and $TC(\phi_{2m+1}) := \xi_m^{(2m+1)} \neq 0$, then there are special Christoffel-Darboux formulae for the OLPs (see, for example, [12, 17]; see, also, [25]):

$$\phi_{2m}(\zeta)(z\phi_{2m-1}(z) - \zeta\phi_{2m-1}(\zeta)) - \zeta\phi_{2m-1}(\zeta)(\phi_{2m}(z) - \phi_{2m}(\zeta)) = (z-\zeta)\frac{\xi_{-m}^{(2m)}}{\xi_{-m}^{(2m-1)}} \sum_{j=0}^{2m-1} \phi_j(z)\phi_j(\zeta),$$

$$\phi_{2m}(\zeta)(z\phi_{2m+1}(z) - \zeta\phi_{2m+1}(\zeta)) - \zeta\phi_{2m+1}(\zeta)(\phi_{2m}(z) - \phi_{2m}(\zeta)) = (z-\zeta)\frac{\xi_m^{(2m+1)}}{\xi_m^{(2m)}} \sum_{j=0}^{2m} \phi_j(z)\phi_j(\zeta),$$

where $\phi_{-1}(z) \equiv 0$, and (dividing by $z - \zeta$ and letting $\zeta \to z$)

$$\phi_{2m}(z)\frac{\mathrm{d}}{\mathrm{d}z}(z\phi_{2m-1}(z)) - z\phi_{2m-1}(z)\frac{\mathrm{d}}{\mathrm{d}z}\phi_{2m}(z) = \frac{\xi_{-m}^{(2m)}}{\xi_{-m}^{(2m-1)}} \sum_{j=0}^{2m-1} (\phi_j(z))^2,$$

$$\phi_{2m}(z)\frac{\mathrm{d}}{\mathrm{d}z}(z\phi_{2m+1}(z))-z\phi_{2m+1}(z)\frac{\mathrm{d}}{\mathrm{d}z}\phi_{2m}(z)=\frac{\xi^{(2m+1)}_{m}}{\xi^{(2m)}_{m}}\sum_{j=0}^{2m}(\phi_{j}(z))^{2}.$$

It is convenient to introduce the monic orthogonal Laurent (or L-) polynomials, $\boldsymbol{\pi}_{j}(z)$, $j\in\mathbb{Z}_{0}^{+}$: (i) for $j=2n$, $n\in\mathbb{Z}_{0}^{+}$, with $\boldsymbol{\pi}_{0}(z)\equiv 1$,

$$\boldsymbol{\pi}_{2n}(z):=\phi_{2n}(z)(\xi^{(2n)}_{n})^{-1}=\nu^{(2n)}_{-n}z^{-n}+\cdots+z^{n}, \qquad \nu^{(2n)}_{-n}:=\xi^{(2n)}_{-n}/\xi^{(2n)}_{n}; \tag{1.4}$$

and (ii) for $j=2n+1$, $n\in\mathbb{Z}_{0}^{+}$,

$$\boldsymbol{\pi}_{2n+1}(z):=\phi_{2n+1}(z)(\xi^{(2n+1)}_{-n-1})^{-1}=z^{-n-1}+\cdots+\nu^{(2n+1)}_{n}z^{n}, \qquad \nu^{(2n+1)}_{n}:=\xi^{(2n+1)}_{n}/\xi^{(2n+1)}_{-n-1}. \tag{1.5}$$

The monic orthogonal L-polynomials, $\boldsymbol{\pi}_{j}(z)$, $j\in\mathbb{Z}_{0}^{+}$, possess the following properties:

(1) $\langle\boldsymbol{\pi}_{2n},z^{j}\rangle_{\mathcal{L}}=0$, $j=-n,\dots,n-1$;

(2) $\langle\boldsymbol{\pi}_{2n+1},z^{j}\rangle_{\mathcal{L}}=0$, $j=-n,\dots,n$;

(3) $\langle\boldsymbol{\pi}_{2n},\boldsymbol{\pi}_{2n}\rangle_{\mathcal{L}}=:\|\boldsymbol{\pi}_{2n}(\cdot)\|^{2}_{\mathcal{L}}=(\xi^{(2n)}_{n})^{-2}$, whence $\xi^{(2n)}_{n}=1/\|\boldsymbol{\pi}_{2n}(\cdot)\|_{\mathcal{L}}$ (>0);

(4) $\langle\boldsymbol{\pi}_{2n+1},\boldsymbol{\pi}_{2n+1}\rangle_{\mathcal{L}}=:\|\boldsymbol{\pi}_{2n+1}(\cdot)\|^{2}_{\mathcal{L}}=(\xi^{(2n+1)}_{-n-1})^{-2}$, whence $\xi^{(2n+1)}_{-n-1}=1/\|\boldsymbol{\pi}_{2n+1}(\cdot)\|_{\mathcal{L}}$ (>0).

Furthermore, in terms of the Hankel determinants, $H^{(m)}_{k}$, $(m,k)\in\mathbb{Z}\times\mathbb{N}$, associated with the real-valued, bi-infinite, strong moment sequence $\left\{c_{k}=\int_{\mathbb{R}}s^{k}\mathrm{e}^{-\mathcal{N}V(s)}\,\mathrm{d}s,\ \mathcal{N}\in\mathbb{N}\right\}_{k\in\mathbb{Z}}$, the monic orthogonal L-polynomials, $\boldsymbol{\pi}_{j}(z)$, $j\in\mathbb{Z}_{0}^{+}$, are represented via the following determinantal formulae [10, 11, 15, 17] (see, also, Subsection 2.2, Proposition 2.2.1): for $m\in\mathbb{Z}_{0}^{+}$,

$$\boldsymbol{\pi}_{2m}(z)=\frac{1}{H^{(-2m)}_{2m}}\begin{vmatrix} c_{-2m} & c_{-2m+1} & \cdots & c_{-1} & z^{-m}\\ c_{-2m+1} & c_{-2m+2} & \cdots & c_{0} & z^{-m+1}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ c_{-1} & c_{0} & \cdots & c_{2m-2} & z^{m-1}\\ c_{0} & c_{1} & \cdots & c_{2m-1} & z^{m}\end{vmatrix}, \tag{1.6}$$

and

$$\boldsymbol{\pi}_{2m+1}(z)=-\frac{1}{H^{(-2m)}_{2m+1}}\begin{vmatrix} c_{-2m-1} & c_{-2m} & \cdots & c_{-1} & z^{-m-1}\\ c_{-2m} & c_{-2m+1} & \cdots & c_{0} & z^{-m}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ c_{-1} & c_{0} & \cdots & c_{2m-1} & z^{m-1}\\ c_{0} & c_{1} & \cdots & c_{2m} & z^{m}\end{vmatrix}; \tag{1.7}$$

moreover, it can be shown that (see, for example, [15, 17]), for $n\in\mathbb{Z}_{0}^{+}$,

$$\xi^{(2n)}_{n}\left(=\frac{1}{\|\boldsymbol{\pi}_{2n}(\cdot)\|_{\mathcal{L}}}\right)=\sqrt{\frac{H^{(-2n)}_{2n}}{H^{(-2n)}_{2n+1}}}, \qquad \xi^{(2n+1)}_{-n-1}\left(=\frac{1}{\|\boldsymbol{\pi}_{2n+1}(\cdot)\|_{\mathcal{L}}}\right)=\sqrt{\frac{H^{(-2n)}_{2n+1}}{H^{(-2n-2)}_{2n+2}}}, \tag{1.8}$$

$$\nu^{(2n)}_{-n}\left(:=\frac{\xi^{(2n)}_{-n}}{\xi^{(2n)}_{n}}\right)=\frac{H^{(-2n+1)}_{2n}}{H^{(-2n)}_{2n}}, \qquad \nu^{(2n+1)}_{n}\left(:=\frac{\xi^{(2n+1)}_{n}}{\xi^{(2n+1)}_{-n-1}}\right)=-\frac{H^{(-2n-1)}_{2n+1}}{H^{(-2n)}_{2n+1}}. \tag{1.9}$$

For each $m\in\mathbb{Z}_{0}^{+}$, the monic orthogonal L-polynomial $\boldsymbol{\pi}_{m}(z)$ and the index m are called *non-singular* if $0\neq TC(\boldsymbol{\pi}_{m}):=\begin{cases}\nu^{(2n)}_{-n}, & m=2n,\\ \nu^{(2n+1)}_{n}, & m=2n+1;\end{cases}$ otherwise, $\boldsymbol{\pi}_{m}(z)$ and m are *singular*. From Equations (1.9), it can be seen that, for each $m\in\mathbb{Z}_{0}^{+}$:

(i) $\boldsymbol{\pi}_{2m}(z)$ is non-singular (resp., singular) if $H^{(-2m+1)}_{2m}\neq 0$ (resp., $H^{(-2m+1)}_{2m}=0$);

(ii) $\boldsymbol{\pi}_{2m+1}(z)$ is non-singular (resp., singular) if $H_{2m+1}^{(-2m-1)} \neq 0$ (resp., $H_{2m+1}^{(-2m-1)}=0$).

For each $m \in \mathbb{Z}_0^+$, let $\mu_{2m}:=\mathrm{card}\{z;\ \boldsymbol{\pi}_{2m}(z)=0\}$ and $\mu_{2m+1}:=\mathrm{card}\{z;\ \boldsymbol{\pi}_{2m+1}(z)=0\}$. It is an established fact [10, 11, 17] that, for $m \in \mathbb{Z}_0^+$:

(1) the zeros of $\boldsymbol{\pi}_{2m}(z)$ are real, simple, and non-zero, and $\mu_{2m}=2m$ (resp., $2m-1$) if $\boldsymbol{\pi}_{2m}(z)$ is non-singular (resp., singular);

(2) the zeros of $\boldsymbol{\pi}_{2m+1}(z)$ are real, simple, and non-zero, and $\mu_{2m+1}=2m+1$ (resp., $2m$) if $\boldsymbol{\pi}_{2m+1}(z)$ is non-singular (resp., singular).

For each $m \in \mathbb{Z}_0^+$, it can be shown that, via a straightforward factorisation argument and using Equations (1.6) and (1.7):

(i) if $\boldsymbol{\pi}_{2m}(z)$ is non-singular, upon setting $\left\{\alpha_k^{(2m)},\ k=1,\ldots,2m\right\}:=\{z;\ \boldsymbol{\pi}_{2m}(z)=0\}$,

$$\prod_{k=1}^{2m} \alpha_k^{(2m)} = \nu_{-m}^{(2m)};$$

(ii) if $\boldsymbol{\pi}_{2m+1}(z)$ is non-singular, upon setting $\left\{\alpha_k^{(2m+1)},\ k=1,\ldots,2m+1\right\}:=\{z;\ \boldsymbol{\pi}_{2m+1}(z)=0\}$,

$$\prod_{k=1}^{2m+1} \alpha_k^{(2m+1)} = -\left(\nu_m^{(2m+1)}\right)^{-1}.$$

Unlike orthogonal polynomials, which satisfy a system of three-term recurrence relations, monic orthogonal, and orthonormal, L-polynomials may satisfy recurrence relations consisting of a pair of four-term recurrence relations [15], a pair of systems of three- or five-term recurrence relations (which is guaranteed in the case when the corresponding monic orthogonal, and orthonormal, L-polynomials are non-singular) [15–17], or a system consisting of four five-term recurrence relations [23].

Remark 1.2. The non-vanishing of the leading and trailing coefficients of the OLPs $\{\phi_m(z)\}_{m=0}^{\infty}$, that is,

$$LC(\phi_m):=\begin{cases} \xi_n^{(2n)}, & m=2n, \\ \xi_{-n-1}^{(2n+1)}, & m=2n+1, \end{cases}$$

and

$$TC(\phi_m):=\begin{cases} \xi_{-n}^{(2n)}, & m=2n, \\ \xi_n^{(2n+1)}, & m=2n+1, \end{cases}$$

respectively, is of paramount importance: if both these conditions are not satisfied, then the 'length' of the recurrence relations may be greater than three [16] (see, also, [24]). ■

It can be shown that (see, for example, [17], and Chapter 11 of [26]), if $\{\boldsymbol{\pi}_m(z)\}_{m \in \mathbb{Z}_0^+}$, as defined above, is a non-singular, monic orthogonal L-polynomial sequence, that is, $H_{2n}^{(-2n+1)} \neq 0$ $(m=2n)$ and $H_{2n+1}^{(-2n-1)} \neq 0$ $(m=2n+1)$, then $\{\boldsymbol{\pi}_m(z)\}_{m \in \mathbb{Z}_0^+}$ satisfy the pair of three-term recurrence relations

$$\begin{aligned} \boldsymbol{\pi}_{2m+1}(z) &= \left(\frac{z^{-1}}{\beta_{2m}^{\natural}} + \beta_{2m+1}^{\natural}\right)\boldsymbol{\pi}_{2m}(z) + \lambda_{2m+1}^{\natural}\boldsymbol{\pi}_{2m-1}(z), \\ \boldsymbol{\pi}_{2m+2}(z) &= \left(\frac{z}{\beta_{2m+1}^{\natural}} + \beta_{2m+2}^{\natural}\right)\boldsymbol{\pi}_{2m+1}(z) + \lambda_{2m+2}^{\natural}\boldsymbol{\pi}_{2m}(z), \end{aligned}$$

where $\boldsymbol{\pi}_{-1}(z) \equiv 0$,

$$\beta_{2m}^{\natural} = \nu_{-m}^{(2m)}, \qquad \beta_{2m+1}^{\natural} = \nu_m^{(2m+1)},$$

$$\lambda_{2m+1}^{\natural} = -\frac{H_{2m+1}^{(-2m-1)} H_{2m-1}^{(-2m+2)}}{H_{2m}^{(-2m)} H_{2m}^{(-2m+1)}} \quad (\neq 0), \qquad \lambda_{2m+2}^{\natural} = -\frac{H_{2m+2}^{(-2m-1)} H_{2m}^{(-2m)}}{H_{2m+1}^{(-2m)} H_{2m+1}^{(-2m-1)}} \quad (\neq 0),$$

and $\lambda_j\beta_{j-1}/\beta_j>0 \ \forall \ j\in\mathbb{N}$, with $\lambda_1:=-c_{-1}$, leading to a *tri-diagonal-type Laurent-Jacobi matrix* $\mathcal{F}$ for the 'mixed' mapping

$$\mathfrak{F}\colon \Lambda^{\mathbb{R}}\to\Lambda^{\mathbb{R}}, \ \ f(z)\mapsto(z^{-1}(\oplus_{n=0}^{\infty}\operatorname{diag}(1,0))+z(\oplus_{n=0}^{\infty}\operatorname{diag}(0,1)))f(z),$$

where $\oplus_{n=0}^{\infty}\operatorname{diag}(1,0):=\operatorname{diag}(1,0,\dots,1,0,\dots)$, and $\oplus_{n=0}^{\infty}\operatorname{diag}(0,1):=\operatorname{diag}(0,1,\dots,0,1,\dots)$,

$$\mathcal{F}=\operatorname{diag}\left(\beta_0^{\natural},\beta_1^{\natural},\beta_2^{\natural},\dots\right)\begin{pmatrix}
-\beta_1^{\natural} & 1 & & & & & & & \\
-\lambda_2^{\natural} & -\beta_2^{\natural} & 1 & & & & & & \\
 & -\lambda_3^{\natural} & -\beta_3^{\natural} & 1 & & & & & \\
 & & -\lambda_4^{\natural} & -\beta_4^{\natural} & 1 & & & & \\
 & & & -\lambda_5^{\natural} & -\beta_5^{\natural} & 1 & & & \\
 & & & & -\lambda_6^{\natural} & -\beta_6^{\natural} & 1 & & \\
 & & & & & \ddots & \ddots & \ddots & \\
 & & & & & -\lambda_{2m+1}^{\natural} & -\beta_{2m+1}^{\natural} & 1 & \\
 & & & & & & -\lambda_{2m+2}^{\natural} & -\beta_{2m+2}^{\natural} & 1 \\
 & & & & & & & \ddots & \ddots \ \ddots
\end{pmatrix},$$

with zeros outside the indicated diagonals (in terms of $\{\phi_m(z)\}_{m\in\mathbb{Z}_0^+}$, the pair of three-term recurrence relations reads [16]:

$$\begin{gathered}\phi_{2m+1}(z)=(z^{-1}+\mathfrak{g}_{2m+1})\phi_{2m}(z)+\mathfrak{f}_{2m+1}\phi_{2m-1}(z),\\ \phi_{2m+2}(z)=(1+\mathfrak{g}_{2m+2}z)\phi_{2m+1}(z)+\mathfrak{f}_{2m+2}\phi_{2m}(z),\end{gathered}$$

where $\mathfrak{f}_{2m+1},\mathfrak{f}_{2m+2}\neq0$, $m\in\mathbb{Z}_0^+$, $\phi_{-1}(z)\equiv0$, and $\phi_0(z)\equiv1$); otherwise, $\{\boldsymbol{\pi}_m(z)\}_{m\in\mathbb{Z}_0^+}$ satisfy the following pair of five-term recurrence relations [17], with $\boldsymbol{\pi}_{-j}(z)\equiv0$, $j=1,2$,

$$\begin{aligned}\boldsymbol{\pi}_{2m+2}(z)&=\gamma^{\flat}_{2m+2,2m-2}\boldsymbol{\pi}_{2m-2}(z)+\gamma^{\flat}_{2m+2,2m-1}\boldsymbol{\pi}_{2m-1}(z)+(z+\gamma^{\flat}_{2m+2,2m})\boldsymbol{\pi}_{2m}(z)\\&\quad+\gamma^{\flat}_{2m+2,2m+1}\boldsymbol{\pi}_{2m+1}(z),\\ \boldsymbol{\pi}_{2m+3}(z)&=\gamma^{\flat}_{2m+3,2m-1}\boldsymbol{\pi}_{2m-1}(z)+\gamma^{\flat}_{2m+3,2m}\boldsymbol{\pi}_{2m}(z)+(z^{-1}+\gamma^{\flat}_{2m+3,2m+1})\boldsymbol{\pi}_{2m+1}(z)\\&\quad+\gamma^{\flat}_{2m+3,2m+2}\boldsymbol{\pi}_{2m+2}(z),\end{aligned}$$

where $\gamma_{l,k}=0$, $k<0$, $l\geqslant2$, leading to a *penta-diagonal-type Laurent-Jacobi matrix* $\mathcal{G}$ for the 'mixed' mapping

$$\mathcal{G}\colon \Lambda^{\mathbb{R}}\to\Lambda^{\mathbb{R}}, \ \ g(z)\mapsto(z(\oplus_{n=0}^{\infty}\operatorname{diag}(1,0))+z^{-1}(\oplus_{n=0}^{\infty}\operatorname{diag}(0,1)))g(z),$$

$$\mathcal{G}=\begin{pmatrix}
-\gamma^{\flat}_{2,0} & -\gamma^{\flat}_{2,1} & 1 & & & & & & \\
-\gamma^{\flat}_{3,0} & -\gamma^{\flat}_{3,1} & -\gamma^{\flat}_{3,2} & 1 & & & & & \\
-\gamma^{\flat}_{4,0} & -\gamma^{\flat}_{4,1} & -\gamma^{\flat}_{4,2} & -\gamma^{\flat}_{4,3} & 1 & & & & \\
 & -\gamma^{\flat}_{5,1} & -\gamma^{\flat}_{5,2} & -\gamma^{\flat}_{5,3} & -\gamma^{\flat}_{5,4} & 1 & & & \\
 & & -\gamma^{\flat}_{6,2} & -\gamma^{\flat}_{6,3} & -\gamma^{\flat}_{6,4} & -\gamma^{\flat}_{6,5} & 1 & & \\
 & & & \ddots & \ddots & \ddots & \ddots & \ddots & \\
 & & & -\gamma^{\flat}_{2m+2,2m-2} & -\gamma^{\flat}_{2m+2,2m-1} & -\gamma^{\flat}_{2m+2,2m} & -\gamma^{\flat}_{2m+2,2m+1} & 1 & \\
 & & & & -\gamma^{\flat}_{2m+3,2m-1} & -\gamma^{\flat}_{2m+3,2m} & -\gamma^{\flat}_{2m+3,2m+1} & -\gamma^{\flat}_{2m+3,2m+2} & 1 \\
 & & & & \ddots & \ddots & \ddots & \ddots & \ddots
\end{pmatrix},$$

with zeros outside the indicated diagonals. The general form of these (system of) recurrence relations is a pair of three- and five-term recurrence relations [23]: for $n\in\mathbb{Z}_0^+$,

$$\begin{gathered}z\phi_{2n+1}(z)=b^{\sharp}_{2n+1}\phi_{2n}(z)+a^{\sharp}_{2n+1}\phi_{2n+1}(z)+b^{\sharp}_{2n+2}\phi_{2n+2}(z),\\ z\phi_{2n}(z)=c^{\sharp}_{2n}\phi_{2n-2}(z)+b^{\sharp}_{2n}\phi_{2n-1}(z)+a^{\sharp}_{2n}\phi_{2n}(z)+b^{\sharp}_{2n+1}\phi_{2n+1}(z)+c^{\sharp}_{2n+2}\phi_{2n+2}(z),\end{gathered}$$

where all the coefficients are real, $c_0^{\sharp}=b_0^{\sharp}=0$, and $c^{\sharp}_{2k}>0$, $k\in\mathbb{N}$, and

$$z^{-1}\phi_{2n}(z)=\beta^{\sharp}_{2n}\phi_{2n-1}(z)+\alpha^{\sharp}_{2n}\phi_{2n}(z)+\beta^{\sharp}_{2n+1}\phi_{2n+1}(z),$$

$$z^{-1}\phi_{2n+1}(z)=\gamma^{\sharp}_{2n+1}\phi_{2n-1}(z)+\beta^{\sharp}_{2n+1}\phi_{2n}(z)+\alpha^{\sharp}_{2n+1}\phi_{2n+1}(z)+\beta^{\sharp}_{2n+2}\phi_{2n+2}(z)+\gamma^{\sharp}_{2n+3}\phi_{2n+3}(z),$$

where all the coefficients are real, $\beta^{\sharp}_0=\gamma^{\sharp}_1=0$, $\beta^{\sharp}_1>0$, and $\gamma^{\sharp}_{2l+1}>0$, $l\in\mathbb{N}$, leading, respectively, to the real-symmetric, *tri-penta-diagonal-type Laurent-Jacobi matrices*, $\mathcal{J}$ and $\mathcal{K}$, for the mappings

$$\mathfrak{J}\colon \Lambda^{\mathbb{R}}\to\Lambda^{\mathbb{R}},\ j(z)\mapsto zj(z)\qquad\text{and}\qquad \mathfrak{K}\colon \Lambda^{\mathbb{R}}\to\Lambda^{\mathbb{R}},\ k(z)\mapsto z^{-1}k(z),$$

$$\mathcal{J}=\left(\begin{array}{ccccccccccccc}
a^{\sharp}_0 & b^{\sharp}_1 & c^{\sharp}_2 & & & & & & & & & & \\
b^{\sharp}_1 & a^{\sharp}_1 & b^{\sharp}_2 & & & & & & & & & & \\
c^{\sharp}_2 & b^{\sharp}_2 & a^{\sharp}_2 & b^{\sharp}_3 & c^{\sharp}_4 & & & & & & & & \\
 & & b^{\sharp}_3 & a^{\sharp}_3 & b^{\sharp}_4 & & & & & & & & \\
 & & c^{\sharp}_4 & b^{\sharp}_4 & a^{\sharp}_4 & b^{\sharp}_5 & c^{\sharp}_6 & & & & & & \\
 & & & & b^{\sharp}_5 & a^{\sharp}_5 & b^{\sharp}_6 & & & & & & \\
 & & & & c^{\sharp}_6 & b^{\sharp}_6 & a^{\sharp}_6 & b^{\sharp}_7 & c^{\sharp}_8 & & & & \\
 & & & & & & b^{\sharp}_7 & a^{\sharp}_7 & b^{\sharp}_8 & & & & \\
 & & & & & & c^{\sharp}_8 & b^{\sharp}_8 & a^{\sharp}_8 & b^{\sharp}_9 & c^{\sharp}_{10} & & \\
 & & & & & & & & \ddots & \ddots & \ddots & & \\
 & & & & & & & & b^{\sharp}_{2k+1} & a^{\sharp}_{2k+1} & b^{\sharp}_{2k+2} & & \\
 & & & & & & & & c^{\sharp}_{2k+2} & b^{\sharp}_{2k+2} & a^{\sharp}_{2k+2} & b^{\sharp}_{2k+3} & c^{\sharp}_{2k+4} \\
 & & & & & & & & & & \ddots & \ddots & \ddots
\end{array}\right),$$

and

$$\mathcal{K}=\left(\begin{array}{ccccccccccccc}
\alpha^{\sharp}_0 & \beta^{\sharp}_1 & & & & & & & & & & & \\
\beta^{\sharp}_1 & \alpha^{\sharp}_1 & \beta^{\sharp}_2 & \gamma^{\sharp}_3 & & & & & & & & & \\
 & \beta^{\sharp}_2 & \alpha^{\sharp}_2 & \beta^{\sharp}_3 & & & & & & & & & \\
 & \gamma^{\sharp}_3 & \beta^{\sharp}_3 & \alpha^{\sharp}_3 & \beta^{\sharp}_4 & \gamma^{\sharp}_5 & & & & & & & \\
 & & & \beta^{\sharp}_4 & \alpha^{\sharp}_4 & \beta^{\sharp}_5 & & & & & & & \\
 & & & \gamma^{\sharp}_5 & \beta^{\sharp}_5 & \alpha^{\sharp}_5 & \beta^{\sharp}_6 & \gamma^{\sharp}_7 & & & & & \\
 & & & & & \beta^{\sharp}_6 & \alpha^{\sharp}_6 & \beta^{\sharp}_7 & & & & & \\
 & & & & & \gamma^{\sharp}_7 & \beta^{\sharp}_7 & \alpha^{\sharp}_7 & \beta^{\sharp}_8 & \gamma^{\sharp}_9 & & & \\
 & & & & & & \beta^{\sharp}_8 & \alpha^{\sharp}_8 & \beta^{\sharp}_9 & & & & \\
 & & & & & & \ddots & \ddots & \ddots & & & & \\
 & & & & & & & & \gamma^{\sharp}_{2k+1} & \beta^{\sharp}_{2k+1} & \alpha^{\sharp}_{2k+1} & \beta^{\sharp}_{2k+2} & \gamma^{\sharp}_{2k+3} \\
 & & & & & & & & & & \beta^{\sharp}_{2k+2} & \alpha^{\sharp}_{2k+2} & \beta^{\sharp}_{2k+3} \\
 & & & & & & & & & & \ddots & \ddots & \ddots
\end{array}\right),$$

with zeros outside the indicated diagonals; moreover, as shown in [23], $\mathcal{J}$ and $\mathcal{K}$ are formal inverses, that is, $\mathcal{J}\mathcal{K}=\mathcal{K}\mathcal{J}=\operatorname{diag}(1,\ldots,1,\ldots)$ (see, also, [27–31]).

It is convenient at this point to discuss, if only succinctly, a few of the multitudinous applications of L-polynomials (complete details may be found in the indicated references):

(1) as stated at the beginning of the Introduction, L-polynomials are intimately related with the solution of the SSMP and the SHMP. It is important to note [14] that the classical and strong moment problems (SMP, HMP, SSMP, and SHMP) are special cases of a more general theory, where moments corresponding to an arbitrary, countable sequence of (fixed) points are involved (in the classical and strong moment cases, respectively, the points are ∞ repeated and $0,\infty$ cyclically repeated), and where *orthogonal rational functions* [26, 32, 33] play the rôle of orthogonal polynomials and orthogonal Laurent (or L-) polynomials; furthermore, since L-polynomials are rational functions with (fixed) poles at the origin and at the point at infinity, the step towards a more general theory where poles are at arbitrary, but fixed, positions/locations in $\mathbb{C}\cup\{\infty\}$ is natural, with applications to, say, multi-point Padé, and Padé-type, approximants [24, 34–38];

(2) in numerical analysis, the computation of integrals of the form $\int_a^b f(s)\,\mathrm{d}\mu(s)$, where μ is a positive measure on $[a,b]$, and $-\infty \leqslant a < b \leqslant +\infty$, is an important problem. The most familiar quadrature formulae are the so-called Gauss-Christoffel formulae, that is, approximating the integral $\int_a^b f(s)\,\mathrm{d}\mu(s)$ via a weighted-sum-of-products of function values of the form $\sum_{j=1}^n \mathcal{A}_{j,n} f(x_{j,n})$, $n \in \mathbb{N}$, where one chooses for the nodes $\{x_{j,n}\}_{j=1}^n$ the zeros/roots of $\varphi_n(z)$, the polynomial of degree n orthogonal with respect to the inner product $\langle f,g\rangle = \int_a^b f(s)\overline{g(s)}\,\mathrm{d}\mu(s)$, and for the (positive) weights $\{\mathcal{A}_{j,n}\}_{j=1}^n$ the so-called Christoffel numbers [35]. When considering the computation of integrals of the form $\int_{-\pi}^{\pi} g(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\mu(\theta)$, where g is a complex-valued function on the unit circle $\mathbb{D} := \{z \in \mathbb{C};\ |z| = 1\}$ and μ is, say, a positive measure on $[-\pi,\pi]$, in particular, when g is continuous on $\mathbb{D}$, keeping in mind that a function continuous on $\mathbb{D}$ can be uniformly approximated by L-polynomials, it is natural to consider, instead of orthogonal polynomials, Laurent polynomials, which are also related to the associated trigonometric moment problem [35, 39] (see, also, [40]);

(3) for $V\colon \mathbb{R}\setminus\{0\} \to \mathbb{R}$ as described by conditions (V1)–(V3), consider the function $g(z) = \int_{\mathbb{R}}(1+sz)^{-1}\,\mathrm{d}\widetilde{\mu}(s)$, where $\mathrm{d}\widetilde{\mu}(s) = \exp(-\mathcal{N}\,V(s))\,\mathrm{d}s$, $\mathcal{N} \in \mathbb{N}$, which is holomorphic for $z \in \mathbb{C}\setminus\mathbb{R}$, with associated asymptotic expansions

$$g(z) \underset{\mathbb{C}\setminus\mathbb{R}\ni z\to 0}{=} \sum_{m=0}^{\infty}(-1)^m c_m z^m =: L^0(z) \qquad \text{and} \qquad g(z) \underset{\mathbb{C}\setminus\mathbb{R}\ni z\to \infty}{=} -\sum_{m=1}^{\infty}(-1)^m c_{-m} z^{-m} =: L^{\infty}(z),$$

where $c_l = \int_{\mathbb{R}} s^l \mathrm{e}^{-\mathcal{N}\,V(s)}\,\mathrm{d}s$, $l \in \mathbb{Z}$, with respect to the (unbounded) domain $\{z \in \mathbb{C};\ \varepsilon \leqslant |\operatorname{Arg}(z)| \leqslant \pi - \varepsilon\}$, where $\operatorname{Arg}(*)$ denotes the principal argument of $*$, and $\varepsilon > 0$ is sufficiently small. Given the pair of formal power series $(L^0(z), L^{\infty}(z))$, the rational function $P_{k,n}(z)/Q_{k,n}(z)$, where $P_{k,n}(z)$ belongs to the space of all polynomials of degree at most $n-1$, and $Q_{k,n}(z)$ is a polynomial of degree exactly n with $Q_{k,n}(0) \neq 0$, is said to be a $[k/n](z)$ *two-point Padé approximant* to $(L^0(z), L^{\infty}(z))$, $k \in \{0,1,\ldots,2n\}$, if the following conditions are satisfied:

$$L^0(z) - P_{k,n}(z)(Q_{k,n}(z))^{-1} \underset{z\to 0}{=} O(z^k),$$
$$L^{\infty}(z) - P_{k,n}(z)(Q_{k,n}(z))^{-1} \underset{z\to \infty}{=} O\Big((z^{-1})^{2n-k+1}\Big).$$

The 'balanced' situation corresponds to the case when $k = n$, in which case, the two-point Padé approximants are denoted, simply, as $[n/n](z)$. An important, related problem of complex approximation theory is to study the convergence of sequences of two-point Padé approximants constructed from the—formal—pair (of power series) $(L^0(z), L^{\infty}(z))$ to the function $g(z)$ on $\mathbb{C}\setminus\mathbb{R}$; in particular, denoting by $\mathcal{E}_n(z)$ the 'error term' for the $[n/n](z)$ approximant, that is, $\mathcal{E}_n(z) := g(z) - [n/n](z)$, it can be shown that, following [41],

$$\mathcal{E}_n(z) = \Big(\phi_n(-1/z)\Big)^{-1} \int_{\mathbb{R}} \frac{\phi_n(s)\mathrm{e}^{-\mathcal{N}\,V(s)}}{1+sz}\,\mathrm{d}s, \quad z \in \mathbb{C}\setminus\mathbb{R}, \tag{TPA1}$$

where $\{\phi_m(z)\}_{m\in\mathbb{Z}_0^+}$ are the orthonormal L-polynomials defined in Equations (1.2) and (1.3). The main question regarding the convergence of two-point Padé approximants for this class of functions is with which rate it takes place, that is, the so-called *quantitative result* [42]: this necessitates obtaining results for the asymptotic behaviour (as $n \to \infty$) of the orthonormal L-polynomials $\phi_n(z)$ in the entire complex plane. The theory of orthogonal L-polynomials is a natural framework for developing the theory of two-point Padé approximants, for both the scalar and matrix cases [24, 41–44];

(4) it turns out that, unlike the (finite) non-relativistic Toda lattice, whose direct and inverse spectral transform was constructed by Moser [45], and which is based on the theory of orthogonal polynomials and tri-diagonal Jacobi matrices, the direct and inverse scattering transform for the (finite) relativistic Toda lattice, introduced by Ruijsenaars [46], is based on the theory of orthogonal L-polynomials and pairs of bi-diagonal matrices [47] (see, also, [48]); and

(5) for a finite, countable or uncountable index set K, let $\{\varsigma_p,\ p \in K\} \subset \mathbb{C}_+ := \{z \in \mathbb{C};\ \operatorname{Im}(z) > 0\}$, with $\varsigma_p \neq \varsigma_q\ \forall\ p \neq q \in K$, and $\{\varpi_p,\ p \in K\} \subset \mathbb{C}$ be given point sets. A function $\mathfrak{F}(z)$ which is analytic for

$z \in \mathbb{C}_+$, with $\mathrm{Im}(\mathfrak{F}(z)) \geqslant 0$, is called a *Nevanlinna function*. The *Pick-Nevanlinna problem* is: find a Nevanlinna function $\mathfrak{F}(z)$ so that $\mathfrak{F}(\varsigma_p) = \varpi_p \ \forall \ p \in K$. A variant of this problem arises when, for $K = \mathbb{N}$, the points ς_p, $p \in \mathbb{N}$, coalesce into the two points 0 and ∞ (the point at infinity) according to the rule $\varsigma_{2i} = 0$, $i \in \mathbb{N}$, $\varsigma_{2j+1} = \infty$, $j \in \mathbb{Z}_0^+$; then, the corresponding modification of the Pick-Nevanlinna problem is: given the bi-infinite sequence of numbers $\{\check{c}_p\}_{p \in \mathbb{Z}}$, find a Nevanlinna function $\mathfrak{F}(z)$ with the asymptotic expansions $\mathfrak{F}(z) \sim_{z \to \infty} \sum_{k=0}^{\infty} \check{c}_k z^{-k}$ and $\mathfrak{F}(z) \sim_{z \to 0} \sum_{k=1}^{\infty} \check{c}_{-k} z^{k}$ in every angular region $\{z \in \mathbb{C};\ \check{\delta} < \mathrm{Arg}(z) < \pi - \check{\delta}\}$, with $\check{\delta} > 0$. This modified problem is equivalent to the SHMP [49].

Now that the principal objects have been defined, namely, the monic OLPs, $\{\boldsymbol{\pi}_m(z)\}_{m \in \mathbb{Z}_0^+}$, and OLPs, $\{\phi_m(z)\}_{m \in \mathbb{Z}_0^+}$, it is time to state what is actually studied in this work; in fact, this work constitutes the first part of a three-fold series of works devoted to asymptotics in the double-scaling limit as $\mathcal{N}, n \to \infty$ such that $z_o := \mathcal{N}/n = 1 + o(1)$ (the simplified 'notation' $n \to \infty$ will be adopted) of L-polynomials and related quantities. From the discussion above, an understanding of the large-n (asymptotic) behaviour of the L-polynomials, as well as of the coefficients of the respective three- and five-term recurrence relations, is seminal in using the L-polynomials in several, seemingly disparate, applications: the purpose of the present series of works is, precisely, to analyse the $n \to \infty$ behaviour of the L-polynomials $\boldsymbol{\pi}_n(z)$ and $\phi_n(z)$ in $\mathbb{C}$, orthogonal with respect to the varying exponential measure[1] $\mathrm{d}\mu(z) = \exp(-n\widetilde{V}(z))\,\mathrm{d}z$, where $\widetilde{V}(z) := z_o V(z)$, and the ('scaled') external field[2] $\widetilde{V} \colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ satisfies conditions (2.3)–(2.5) (see Subsection 2.2), as well as of the associated norming constants and coefficients of the (system of) recurrence relations; more precisely, then:

- **(i)** in this work (Part I), asymptotics (as $n \to \infty$) of $\boldsymbol{\pi}_{2n}(z)$ (in the entire complex plane) and $\xi_n^{(2n)}$, thus $\phi_{2n}(z)$ (cf. Equation (1.4)), and the Hankel determinant ratio $H_{2n}^{(-2n)}/H_{2n+1}^{(-2n)}$ (cf. Equations (1.8)) are obtained;
- **(ii)** in Part II [51], asymptotics (as $n \to \infty$) of $\boldsymbol{\pi}_{2n+1}(z)$ (in the entire complex plane) and $\xi_{-n-1}^{(2n+1)}$, thus $\phi_{2n+1}(z)$ (cf. Equation (1.5)), and the Hankel determinant ratio $H_{2n+1}^{(-2n)}/H_{2n+2}^{(-2n-2)}$ (cf. Equations (1.8)) are obtained;
- **(iii)** in Part III [52], asymptotics (as $n \to \infty$) of $\nu_{-n}^{(2n)}$ $(= H_{2n}^{(-2n+1)}/H_{2n}^{(-2n)})$ and $\xi_{-n}^{(2n)}$, $\nu_n^{(2n+1)}$ $(= -H_{2n+1}^{(-2n-1)}/H_{2n+1}^{(-2n)})$ and $\xi_n^{(2n+1)}$, $\prod_{k=1}^{2n} \alpha_k^{(2n)}$ $(= \nu_{-n}^{(2n)})$, and $\prod_{k=1}^{2n+1} \alpha_k^{(2n+1)}$ $(= -(\nu_n^{(2n+1)})^{-1})$, as well as of the (elements of the) Laurent-Jacobi matrices, $\mathcal{J}$ and $\mathcal{K}$, and other, related, quantities constructed from the coefficients of the three- and five-term recurrence relations, are obtained.

The above-mentioned asymptotics (as $n \to \infty$) are obtained by reformulating, *à la* Fokas-Its-Kitaev [53, 54], the corresponding even degree and odd degree monic L-polynomial problems as (matrix) Riemann-Hilbert problems (RHPs) on $\mathbb{R}$, and then studying the large-n behaviour of the corresponding solutions. The paradigm for the asymptotic (as $n \to \infty$) analysis of the respective (matrix) RHPs is a union of the Deift-Zhou (DZ) non-linear steepest-descent method [1, 2], used for the asymptotic analysis of undulatory RHPs, and the extension of Deift-Venakides-Zhou [3], incorporating into the DZ method a non-linear analogue of the WKB method, making the asymptotic analysis of fully non-linear problems tractable (it should be mentioned that, in this context, the equilibrium measure [55] plays an absolutely crucial rôle in the analysis [56]); see, also, the multitudinous extensions and applications of the DZ method [57–79]. It is worth mentioning that asymptotics for Laurent-type polynomials and their zeros have been obtained in [42, 80] (see, also, [81–83]).

This article is organised as follows. In Section 2, necessary facts from the theory of compact Riemann surfaces are given, the respective 'even degree' and 'odd degree' RHPs on $\mathbb{R}$ are stated and the corresponding variational problems for the associated equilibrium measures are discussed, and the main results of this work, namely, asymptotics (as $n \to \infty$) of $\boldsymbol{\pi}_{2n}(z)$ (in $\mathbb{C}$), and $\xi_n^{(2n)}$ and $\phi_{2n}(z)$ (in $\mathbb{C}$) are stated in Theorems 2.3.1 and 2.3.2, respectively. In Section 3, the detailed analysis of the 'even degree' variational problem and the associated equilibrium measure is undertaken, including

[1]Note that $LD(\boldsymbol{\pi}_m) = LD(\phi_m) = \begin{cases} 2n, & m = \text{even}, \\ 2n+1, & m = \text{odd}, \end{cases}$ coincides with the parameter in the measure of orthogonality: the large parameter, n, enters simultaneously into the L-degree of the L-polynomials and the (varying exponential) weight; thus, asymptotics of the L-polynomials are studied along a 'diagonal strip' of a doubly-indexed sequence.

[2]For real non-analytic external fields, see the recent work [50].

the construction of the so-called g-function, and the RHP formulated in Section 2 is reformulated as an equivalent, auxiliary RHP, which, in Sections 4 and 5, is augmented, by means of a sequence of contour deformations and transformations *à la* Deift-Venakides-Zhou, into simpler, 'model' (matrix) RHPs which, as $n \to \infty$, and in conjunction with the Beals-Coifman construction [84] (see, also, the extension of Zhou [85]) for the integral representation of the solution of a matrix RHP on an oriented contour, are solved explicitly (in closed form) in terms of Riemann theta functions (associated with the underlying finite-genus hyperelliptic Riemann surface) and Airy functions, from which the final asymptotic (as $n \to \infty$) results stated in Theorems 2.3.1 and 2.3.2 are proved. The paper concludes with an Appendix.

Remark 1.3. The even degree OLPs, $\phi_{2n}(z)$, $n \in \mathbb{Z}_0^+$, are related, in a way, to the polynomials orthogonal with respect to the varying weight $\widetilde{w}(z) := z^{-2n}\exp(-\mathcal{N}\,V(z))$, $\mathcal{N} \in \mathbb{N}$: this follows directly from the orthogonality relation satisfied by $\phi_{2n}(z)$. This does not help with any of the algebraic relations, such as the system of three- and five-term recurrence relations; however, this does provide for an alternative approach to computing large-n asymptotics for $\phi_{2n}(z)$. The connection is not so clear for the odd degree OLPs, $\phi_{2n+1}(z)$, $n \in \mathbb{Z}_0^+$. Indeed, in this latter case, the associated (density of the) measure for the orthogonal polynomials would take the form $\mathrm{d}\widehat{\mu}(z) := z^{-2n-1}\exp(-\mathcal{N}\,V(z))\,\mathrm{d}z$, and this measure changes signs, which causes a number of difficulties in the large-n asymptotic analysis. In this paper, these connections are not used, and a complete asymptotic analysis of the even degree OLPs is carried out, directly. ■

2 Hyperelliptic Riemann Surfaces, The Riemann-Hilbert Problems, and Summary of Results

In this section, necessary facts from the theory of hyperelliptic Riemann surfaces are given (see Subsection 2.1), the respective RHPs on $\mathbb{R}$ for the even degree and odd degree monic orthogonal L-polynomials are formulated and the corresponding variational problems for the associated equilibrium measures are discussed (see Subsection 2.2), and asymptotics (as $n \to \infty$) for $\boldsymbol{\pi}_{2n}(z)$ (in the entire complex plane), and $\xi_n^{(2n)}$ and $\phi_{2n}(z)$ (in the entire complex plane) are given in Theorems 2.3.1 and 2.3.2, respectively (see Subsection 2.3).

Before proceeding, however, the notation/nomenclature used throughout this work is summarised.

Notational Conventions

(1) $\mathrm{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the 2×2 identity matrix, $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$ and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are the Pauli matrices, $\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ are, respectively, the raising and lowering matrices, $\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\mathbb{R}_\pm := \{x \in \mathbb{R};\ \pm x > 0\}$, $\mathbb{C}_\pm := \{z \in \mathbb{C};\ \pm\mathrm{Im}(z) > 0\}$, and $\mathrm{sgn}(x) := 0$ if $x = 0$ and $x|x|^{-1}$ if $x \neq 0$;

(2) for a scalar ω and a 2×2 matrix Υ, $\omega^{\mathrm{ad}(\sigma_3)}\Upsilon := \omega^{\sigma_3}\Upsilon\omega^{-\sigma_3}$;

(3) a contour $\mathcal{D}$ which is the finite union of piecewise-smooth, simple curves (as closed sets) is said to be *orientable* if its complement $\mathbb{C} \setminus \mathcal{D}$ can always be divided into two, possibly disconnected, disjoint open sets $\mho^+$ and $\mho^-$, either of which has finitely many components, such that $\mathcal{D}$ admits an orientation so that it can either be viewed as a positively oriented boundary $\mathcal{D}^+$ for $\mho^+$ or as a negatively oriented boundary $\mathcal{D}^-$ for $\mho^-$ [85], that is, the (possibly disconnected) components of $\mathbb{C} \setminus \mathcal{D}$ can be coloured by $+$ or $-$ in such a way that the $+$ regions do not share boundary with the $-$ regions, except, possibly, at finitely many points [86];

(4) for each segment of an oriented contour $\mathcal{D}$, according to the given orientation, the "+" side is to the left and the "-" side is to the right as one traverses the contour in the direction of orientation, that is, for a matrix $\mathcal{A}_{ij}(z)$, $i, j = 1, 2$, $(\mathcal{A}_{ij}(z))_\pm$ denote the non-tangential limits $(\mathcal{A}_{ij}(z))_\pm := \lim_{\substack{z' \to z \\ z' \in \pm \text{ side of } \mathcal{D}}} \mathcal{A}_{ij}(z')$;

(5) for $1 \leqslant p < \infty$ and $\mathcal{D}$ some point set,

$$\mathcal{L}^p_{\mathrm{M}_2(\mathbb{C})}(\mathcal{D}) := \left\{ f\colon \mathcal{D} \to \mathrm{M}_2(\mathbb{C});\ \|f(\cdot)\|_{\mathcal{L}^p_{\mathrm{M}_2(\mathbb{C})}(\mathcal{D})} := \left(\int_{\mathcal{D}} |f(z)|^p \,|\mathrm{d}z| \right)^{1/p} < \infty \right\},$$

where, for $\mathcal{A}(z) \in \mathrm{M}_2(\mathbb{C})$, $|\mathcal{A}(z)| := (\sum_{i,j=1}^{2} \overline{\mathcal{A}_{ij}(z)}\,\mathcal{A}_{ij}(z))^{1/2}$ is the Hilbert-Schmidt norm, with $\overline{\bullet}$ denoting complex conjugation of $\bullet$, for $p=\infty$,

$$\mathcal{L}^{\infty}_{\mathrm{M}_2(\mathbb{C})}(\mathcal{D}) := \left\{ g\colon \mathcal{D} \to \mathrm{M}_2(\mathbb{C});\ \|g(\cdot)\|_{\mathcal{L}^{\infty}_{\mathrm{M}_2(\mathbb{C})}(\mathcal{D})} := \max_{i,j=1,2}\ \sup_{z\in\mathcal{D}} |g_{ij}(z)| < \infty \right\},$$

and, for $f \in \mathrm{I} + \mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\mathcal{D}) := \left\{ \mathrm{I}+h;\ h \in \mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\mathcal{D}) \right\}$,

$$\|f(\cdot)\|_{\mathrm{I}+\mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\mathcal{D})} := \left(\|f(\infty)\|^2_{\mathcal{L}^{\infty}_{\mathrm{M}_2(\mathbb{C})}(\mathcal{D})} + \|f(\cdot)-f(\infty)\|^2_{\mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\mathcal{D})} \right)^{1/2};$$

(6) for a matrix $\mathcal{A}_{ij}(z)$, $i,j=1,2$, to have boundary values in the $\mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\mathcal{D})$ sense on an oriented contour $\mathcal{D}$, it is meant that $\lim_{\substack{z' \to z \\ z' \in \pm \text{ side of } \mathcal{D}}} \int_{\mathcal{D}} |\mathcal{A}(z') - (\mathcal{A}(z))_{\pm}|^2\,|\mathrm{d}z| = 0$ (e.g., if $\mathcal{D} = \mathbb{R}$ is oriented from $+\infty$ to $-\infty$, then $\mathcal{A}(z)$ has $\mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\mathcal{D})$ boundary values on $\mathcal{D}$ means that $\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}} |\mathcal{A}(x \mp \mathrm{i}\varepsilon) - (\mathcal{A}(x))_{\pm}|^2\,\mathrm{d}x = 0$);

(7) for a 2×2 matrix-valued function $\mathfrak{Y}(z)$, the notation $\mathfrak{Y}(z) =_{z\to z_0} \mathcal{O}(*)$ means $\mathfrak{Y}_{ij}(z) =_{z\to z_0} \mathcal{O}(*_{ij})$, $i,j=1,2$ (*mutatis mutandis* for $o(1)$);

(8) $\|\mathfrak{F}(\cdot)\|_{\cap_{p\in J}\mathcal{L}^{p}_{\mathrm{M}_2(\mathbb{C})}(*)} := \sum_{p\in J} \|\mathfrak{F}(\cdot)\|_{\mathcal{L}^{p}_{\mathrm{M}_2(\mathbb{C})}(*)}$, with $\mathrm{card}(J) < \infty$;

(9) $\mathcal{M}_1(\mathbb{R})$ denotes the set of all non-negative, bounded, unit Borel measures on $\mathbb{R}$ for which all moments exist,

$$\mathcal{M}_1(\mathbb{R}) := \left\{ \mu;\ \int_{\mathbb{R}} \mathrm{d}\mu(s) = 1,\ \int_{\mathbb{R}} s^m\,\mathrm{d}\mu(s) < \infty,\ m \in \mathbb{Z}\setminus\{0\} \right\};$$

(10) for $(\mu,\nu) \in \mathbb{R}\times\mathbb{R}$, denote the function $(\bullet-\mu)^{\mathrm{i}\nu}\colon \mathbb{C}\setminus(-\infty,\mu) \to \mathbb{C}$, $\bullet \mapsto \exp(\mathrm{i}\nu\ln(\bullet-\mu))$, where ln denotes the principal branch of the logarithm;

(11) for $\widetilde{\gamma}$ a null-homologous path in a region $\mathfrak{D}\subset\mathbb{C}$, $\mathrm{int}(\widetilde{\gamma}) := \left\{ \zeta \in \mathfrak{D}\setminus\widetilde{\gamma};\ \mathrm{ind}_{\widetilde{\gamma}}(\zeta) := \int_{\widetilde{\gamma}} \frac{1}{z-\zeta}\frac{\mathrm{d}z}{2\pi\mathrm{i}} \neq 0 \right\}$;

(12) for some point set $\mathcal{D}\subset\mathcal{X}$, with $\mathcal{X}=\mathbb{C}$ or $\mathbb{R}$, $\overline{\mathcal{D}} := \mathcal{D}\cup\partial\mathcal{D}$, and $\mathcal{D}^c := \mathcal{X}\setminus\overline{\mathcal{D}}$.

2.1 Riemann Surfaces: Preliminaries

In this subsection, the basic elements associated with the construction of hyperelliptic and finite genus (compact) Riemann surfaces are presented (for further details and proofs, see, for example, [87, 88]).

Remark 2.1.1. The superscripts $^{\pm}$, and sometimes subscripts $_{\pm}$, in this subsection should not be confused with the subscripts $_{\pm}$ appearing in the various RHPs (this is a general comment which applies, unless stated otherwise, throughout the entire text). Although $\overline{\mathbb{C}}$ (or $\mathbb{CP}^1$) $:= \mathbb{C}\cup\{\infty\}$ (resp., $\overline{\mathbb{R}} := \mathbb{R}\cup\{-\infty\}\cup\{+\infty\}$) is the standard definition for the (closed) Riemann sphere (resp., closed real line), the simplified, and somewhat abusive, notation $\mathbb{C}$ (resp., $\mathbb{R}$) is used to denote both the (closed) Riemann sphere, $\overline{\mathbb{C}}$ (resp., closed real line, $\overline{\mathbb{R}}$), and the (open) complex field, $\mathbb{C}$ (resp., open real line, $\mathbb{R}$), and the context(s) should make clear which object(s) the notation $\mathbb{C}$ (resp., $\mathbb{R}$) represents. ∎

Let $N \in \mathbb{N}$ (with $N < \infty$ assumed throughout) and $\varsigma_k \in \mathbb{R}\setminus\{0,\pm\infty\}$, $k = 1,\dots,2N+2$, be such that $\varsigma_i \neq \varsigma_j$ $\forall$ $i \neq j = 1,\dots,2N+2$, and enumerated/ordered according to $\varsigma_1 < \varsigma_2 < \cdots < \varsigma_{2N+2}$. Let $R(z) := \prod_{j=1}^{N}(z-\varsigma_{2j-1})(z-\varsigma_{2j}) \in \mathbb{R}[z]$ (the algebra of polynomials in z with coefficients in $\mathbb{R}$) be the (unital) polynomial of even degree $\deg(R) = 2N+2$ ($\deg(R) = 0 \pmod 2$) whose (simple) zeros/roots are $\{\varsigma_j\}_{j=1}^{2N+2}$. Denote by $\mathcal{R}$ the hyperelliptic Riemann surface of genus N defined by the equation $y^2 = R(z)$ and realised as a two-sheeted branched (ramified) covering of the Riemann sphere such that its two sheets are two identical copies of $\mathbb{C}$ with branch cuts along the intervals $(\varsigma_1,\varsigma_2), (\varsigma_3,\varsigma_4), \dots, (\varsigma_{2N+1},\varsigma_{2N+2})$, and glued/pasted to each other 'crosswise' along the opposite banks of the corresponding cuts $(\varsigma_{2j-1},\varsigma_{2j})$, $j = 1,\dots,N+1$. Denote the two sheets of $\mathcal{R}$ by $\mathcal{R}^+$ (the first/upper sheet) and $\mathcal{R}^-$ (the second/lower sheet): to indicate that z lies on the first (resp., second) sheet, one writes z^+ (resp., z^-); of course, as points in the plane $\mathbb{C}$, $z^+ = z^- = z$. For points z on the first (resp., second) sheet $\mathcal{R}^+$ (resp., $\mathcal{R}^-$), one has $z^+ = (z, +(R(z))^{1/2})$ (resp., $z^- = (z, -(R(z))^{1/2})$), where the single-valued branch of the square root is chosen such that $z^{-(N+1)}(R(z))^{1/2} \sim_{\substack{z\to\infty \\ z\in\mathcal{R}^{\pm}}} \pm 1$.

Let $\mathcal{E}_j := (\varsigma_{2j-1}, \varsigma_{2j})$, $j=1,\dots,N+1$, and set $\mathcal{E} = \cup_{j=1}^{N+1}\mathcal{E}_j$ (note that $\mathcal{E}_i \cap \mathcal{E}_j = \varnothing$, $i \neq j = 1,\dots,N+1$). Denote by $\mathcal{E}_j^+$ $(\subset \mathcal{R}^+)$ (resp., $\mathcal{E}_j^-$ $(\subset \mathcal{R}^-)$) the upper (resp., lower) bank of the interval $\mathcal{E}_j$, $j=1,\dots,N+1$, forming $\mathcal{E}$, and oriented in accordance with the orientation of $\mathcal{E}$ as the boundary of $\mathbb{C} \setminus \mathcal{E}$, namely, the domain $\mathbb{C} \setminus \mathcal{E}$ is on the left as one proceeds along the upper bank of the jth interval from ς_{2j-1} to the point ς_{2j} and back along the lower bank from ς_{2j} to ς_{2j-1}; thus, $\mathcal{E}_j^{\pm} := (\varsigma_{2j-1}, \varsigma_{2j})^{\pm}$, $j=1,\dots,N+1$, are two (identical) copies of $(\varsigma_{2j-1}, \varsigma_{2j}) \subset \mathbb{R}$ 'lifted' to $\mathcal{R}^{\pm}$. Set $\Gamma := \cup_{j=1}^{N+1}\Gamma_j$ $(\subset \mathcal{R})$, where $\Gamma_j := \mathcal{E}_j^+ \cup \mathcal{E}_j^-$, $j=1,\dots,N+1$ $(\Gamma = \mathcal{E}^+ \cup \mathcal{E}^-)$: note that Γ, as a curve on $\mathcal{R}$ (defined by the equation $y^2 = R(z)$), consists of a finitely denumerable number of disjoint analytic closed Jordan curves, Γ_j, $j=1,\dots,N+1$, which are *cycles* on $\mathcal{R}$, and that correspond to the intervals $\mathcal{E}_j$. From the above construction, it is clear that $\mathcal{R} = \mathcal{R}^+ \cup \mathcal{R}^- \cup \Gamma$; furthermore, the canonical projection of Γ onto $\mathbb{C}$ ($\boldsymbol{\pi}\colon \mathcal{R} \to \mathbb{C}$) is $\mathcal{E}$, that is, $\boldsymbol{\pi}(\Gamma) = \mathcal{E}$ (also, $\boldsymbol{\pi}(\mathcal{R}^+) = \boldsymbol{\pi}(\mathcal{R}^-) = \mathbb{C} \setminus \mathcal{E}$, or, alternately, $\boldsymbol{\pi}(z^+) = \boldsymbol{\pi}(z^-) = z$). One moves in the 'positive (+)' (resp., 'negative (−)') direction along the (closed) contour $\Gamma \subset \mathcal{R}$ if the domain $\mathcal{R}^+$ is on the left (resp., right) and the domain $\mathcal{R}^-$ is on the right (resp., left): the corresponding notation is (see above) Γ^+ (resp., Γ^-). For a function f defined on the two-sheeted hyperelliptic Riemann surface $\mathcal{R}$, one defines the non-tangential boundary values, provided they exist, of $f(z)$ as $z \in \mathcal{R}^+$ (resp., $z \in \mathcal{R}^-$) approaches $\lambda \in \Gamma$, denoted λ_+ (resp., λ_-), by $f(\lambda_{\pm}) := f_{\pm}(\lambda) := \lim_{\substack{z \to \lambda \\ z \in \Gamma^{\pm}}} f(z)$.

One takes the first N contours among the (closed) contours Γ_j for basis $\boldsymbol{\alpha}$-cycles $\{\boldsymbol{\alpha}_j,\ j=1,\dots,N\}$ and then completes/supplements this in the standard way with $\boldsymbol{\beta}$-cycles $\{\boldsymbol{\beta}_j,\ j=1,\dots,N\}$ so that the *intersection matrix* has the (canonical) form $\boldsymbol{\alpha}_k \circ \boldsymbol{\alpha}_j = \boldsymbol{\beta}_k \circ \boldsymbol{\beta}_j = 0$ $\forall$ $k \neq j = 1,\dots,N$, and $\boldsymbol{\alpha}_k \circ \boldsymbol{\beta}_j = \delta_{kj}$: the cycles $\{\boldsymbol{\alpha}_j, \boldsymbol{\beta}_j\}$, $j=1,\dots,N$, form the *canonical 1-homology basis* on $\mathcal{R}$, namely, any cycle $\widehat{\boldsymbol{\gamma}} \subset \mathcal{R}$ is homologous to an integral linear combination of $\{\boldsymbol{\alpha}_j, \boldsymbol{\beta}_j\}$, that is, $\widehat{\boldsymbol{\gamma}} = \sum_{j=1}^{N}(n_j\boldsymbol{\alpha}_j + m_j\boldsymbol{\beta}_j)$, where $(n_j, m_j) \in \mathbb{Z} \times \mathbb{Z}$, $j=1,\dots,N$. The $\boldsymbol{\alpha}$-cycles $\{\boldsymbol{\alpha}_j,\ j=1,\dots,N\}$, in the present case, are the intervals $(\varsigma_{2j-1}, \varsigma_{2j})$, $j=1,\dots,N$, 'going twice', that is, along the upper (from ς_{2j-1} to ς_{2j}) and lower (from ς_{2j} to ς_{2j-1}) banks ($\alpha_j = \mathcal{E}_j^+ \cup \mathcal{E}_j^-$, $j=1,\dots,N$), and the $\boldsymbol{\beta}$-cycles $\{\boldsymbol{\beta}_j,\ j=1,\dots,N\}$ are as follows: the jth $\boldsymbol{\beta}$-cycle consists of the $\boldsymbol{\alpha}$-cycles $\boldsymbol{\alpha}_k$, $k=j+1,\dots,N$, and the cycles 'linked' with them and consisting of (the gaps) $(\varsigma_{2k}, \varsigma_{2k+1})$, $k=1,\dots,N$, 'going twice', that is, from ς_{2k} to ς_{2k+1} on the first sheet and in the reverse direction on the second sheet. For an arbitrary holomorphic Abelian differential (one-form) $\boldsymbol{\omega}$ on $\mathcal{R}$, the function $\int^z \boldsymbol{\omega}$ is defined uniquely modulo its $\boldsymbol{\alpha}$- and $\boldsymbol{\beta}$-periods, $\oint_{\boldsymbol{\alpha}_j} \boldsymbol{\omega}$ and $\oint_{\boldsymbol{\beta}_j} \boldsymbol{\omega}$, $j=1,\dots,N$, respectively. It is well known that the canonical 1-homology basis $\{\boldsymbol{\alpha}_j, \boldsymbol{\beta}_j\}$, $j=1,\dots,N$, constructed above 'generates', on $\mathcal{R}$, the corresponding $\boldsymbol{\alpha}$-normalised basis of holomorphic Abelian differentials (one-forms) $\{\omega_1, \omega_2, \dots, \omega_N\}$, where $\omega_j := \sum_{k=1}^{N} \frac{c_{jk}z^{N-k}}{\sqrt{R(z)}}\,\mathrm{d}z$, $c_{jk} \in \mathbb{C}$, $j=1,\dots,N$, and $\oint_{\boldsymbol{\alpha}_k} \omega_j = \delta_{kj}$, $k,\ j=1,\dots,N$: the associated $N \times N$ matrix of $\boldsymbol{\beta}$-periods, $\tau = (\tau_{ij})_{i,j=1,\dots,N} := \left(\oint_{\boldsymbol{\beta}_j} \omega_i\right)_{i,j=1,\dots,N}$, is a *Riemann matrix*, that is, it is symmetric ($\tau_{ij} = \tau_{ji}$), pure imaginary, and $-\mathrm{i}\tau$ is positive definite ($\mathrm{Im}(\tau_{jj}) > 0$); moreover, τ is non-degenerate ($\det(\tau) \neq 0$). From the condition that the basis of the differentials ω_l, $l=1,\dots,N$, is canonical, with respect to the given basis cycles $\{\boldsymbol{\alpha}_j, \boldsymbol{\beta}_j\}$, it is seen that this implies that each ω_l is real valued on $\mathcal{E} = \cup_{j=1}^{N+1}(\varsigma_{2j-1}, \varsigma_{2j})$ and has exactly one (real) root/zero in any interval (band) $(\varsigma_{2j-1}, \varsigma_{2j})$, $j=1,\dots,N+1$, $j \neq l$; moreover, in the 'gaps' $(\varsigma_{2j}, \varsigma_{2j+1})$, $j=1,\dots,N$, these differentials take non-zero, pure imaginary values.

Fix the 'standard basis' $\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_N$ in $\mathbb{R}^N$, that is, $(\boldsymbol{e}_j)_k = \delta_{jk}$, $j,k=1,\dots,N$ (these standard basis vectors should be viewed as column vectors): the vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_N, \tau\boldsymbol{e}_1, \tau\boldsymbol{e}_2, \dots, \tau\boldsymbol{e}_N$ are linearly independent over $\mathbb{R}$, and form a 'basis' in $\mathbb{C}^N$. The quotient space $\mathbb{C}^N/\{N{+}\tau M\}$, $(N,M) \in \mathbb{Z}^N \times \mathbb{Z}^N$, where $\mathbb{Z}^N := \{(m_1, m_2, \dots, m_N);\ m_j \in \mathbb{Z},\ j=1,\dots,N\}$, is a $2N$-dimensional real torus $\mathbb{T}^{2N}$, and is referred to as the *Jacobi variety*, symbolically $\mathrm{Jac}(\mathcal{R})$, of the two-sheeted (hyperelliptic) Riemann surface $\mathcal{R}$ of genus N. Let z_0 be a fixed point in $\mathcal{R}$. A vector-valued function $\mathcal{A}(z) = (\mathcal{A}_1(z), \mathcal{A}_2(z), \dots, \mathcal{A}_N(z)) \in \mathrm{Jac}(\mathcal{R})$ with co-ordinates $\mathcal{A}_k(z) \equiv \int_{z_0}^{z} \omega_k$, $k=1,\dots,N$, where, hereafter, unless stated otherwise and/or where confusion may arise, $\equiv$ denotes 'congruence modulo the period lattice', defines the *Abel map* $\mathcal{A}\colon \mathcal{R} \to \mathrm{Jac}(\mathcal{R})$. The unordered set of points $z_1, z_2, \dots, z_N$, with $z_k \in \mathcal{R}$, form the Nth symmetric power of $\mathcal{R}$, symbolically $\mathcal{R}^N_{\mathrm{symm}}$ (or $\mathcal{S}^N\mathcal{R}$). The vector function $\boldsymbol{\mathfrak{U}} = (\mathfrak{U}_1, \mathfrak{U}_2, \dots, \mathfrak{U}_N)$ with co-ordinates $\mathfrak{U}_j = \mathfrak{U}_j(z_1, z_2, \dots, z_N) \equiv \sum_{k=1}^{N} \mathcal{A}_j(z_k) \equiv \sum_{k=1}^{N} \int_{z_0}^{z_k} \omega_j$, $j=1,\dots,N$, that is, $(z_1, z_2, \dots, z_N) \to (\sum_{k=1}^{N}\int_{z_0}^{z_k}\omega_1, \sum_{k=1}^{N}\int_{z_0}^{z_k}\omega_2, \dots, \sum_{k=1}^{N}\int_{z_0}^{z_k}\omega_N)$, is also referred to as the *Abel map*, $\boldsymbol{\mathfrak{U}}\colon \mathcal{R}^N_{\mathrm{symm}} \to \mathrm{Jac}(\mathcal{R})$ (or $\boldsymbol{\mathfrak{U}}\colon \mathcal{S}^N\mathcal{R} \to \mathrm{Jac}(\mathcal{R})$). It is known (see, for example, [89]) that the Abel map $\boldsymbol{\mathfrak{U}}\colon \mathcal{R}^N_{\mathrm{symm}} \to \mathrm{Jac}(\mathcal{R})$ is surjective and locally biholomorphic, but not injective globally. The *dissected* Riemann surface, symbolically $\widetilde{\mathcal{R}}$, is obtained from $\mathcal{R}$ by 'cutting'

(canonical dissection) along the cycles of the canonical 1-homology basis $\boldsymbol{\alpha}_k, \boldsymbol{\beta}_k$, $k=1,\dots,N$, of the original surface, namely, $\widetilde{\mathcal{R}} = \mathcal{R} \setminus (\cup_{j=1}^{N}(\boldsymbol{\alpha}_j \cup \boldsymbol{\beta}_j))$; the surface $\widetilde{\mathcal{R}}$ is not only connected, as one can 'pass' from one sheet to the other 'across' Γ_{N+1}, but also simply connected (a $4N$-sided polygon ($4N$-gon) of a canonical dissection of $\mathcal{R}$ associated with the given canonical 1-homology basis for $\mathcal{R}$). For a given vector $\vec{\boldsymbol{v}} = (v_1, v_2, \dots, v_N) \in \mathrm{Jac}(\mathcal{R})$, the problem of finding an unordered collection of points $z_1, z_2, \dots, z_N$, $z_j \in \mathcal{R}$, $j=1,\dots,N$, for which $\mathfrak{U}_k(z_1, z_2, \dots, z_N) \equiv v_k$, $k=1,\dots,N$, is called the *Jacobi inversion problem* for Abelian integrals: as is well known, the Jacobi inversion problem is always solvable; but not, in general, uniquely.

By a *divisor* on the Riemann surface $\mathcal{R}$ is meant a formal 'symbol' $\boldsymbol{d} = z_1^{n_f(z_1)} z_2^{n_f(z_2)} \cdots z_m^{n_f(z_m)}$, where $z_j \in \mathcal{R}$ and $n_f(z_j) \in \mathbb{Z}$, $j=1,\dots,m$: the number $|\boldsymbol{d}| := \sum_{j=1}^{m} n_f(z_j)$ is called the *degree* of the divisor $\boldsymbol{d}$: if $z_i \neq z_j$ $\forall$ $i \neq j = 1,\dots,m$, and if $n_f(z_j) \geqslant 0$, $j=1,\dots,m$, then the divisor $\boldsymbol{d}$ is said to be *integral*. Let g be a meromorphic function defined on $\mathcal{R}$: for an arbitrary point $a \in \mathcal{R}$, one denotes by $n_g(a)$ (resp., $p_g(a)$) the multiplicity of the zero (resp., pole) of the function g at this point if a is a zero (resp., pole), and sets $n_g(a)=0$ (resp., $p_g(a)=0$) otherwise; thus, $n_g(a)$, $p_g(a) \geqslant 0$. To a meromorphic function g on $\mathcal{R}$, one assigns the divisor (g) of zeros and poles of this function as $(g) = z_1^{n_g(z_1)} z_2^{n_g(z_2)} \cdots z_{l_1}^{n_g(z_{l_1})} \lambda_1^{-p_g(\lambda_1)} \lambda_2^{-p_g(\lambda_2)} \cdots \lambda_{l_2}^{-p_g(\lambda_{l_2})}$, where z_i, $\lambda_j \in \mathcal{R}$, $i=1,\dots,l_1$, $j=1,\dots,l_2$, are the zeros and poles of g on $\mathcal{R}$, and $n_g(z_i)$, $p_g(\lambda_j) \geqslant 0$ are their multiplicities (one can also write $\{(a, n_g(a), -p_g(a));\ a \in \mathcal{R},\ n_g(a),\ p_g(a) \geqslant 0\}$ for the divisor (g) of g): these divisors are said to be *principal*.

Associated with the Riemann matrix of $\boldsymbol{\beta}$-periods, τ, is the *Riemann theta function*, defined by

$$\boldsymbol{\theta}(z;\tau) =: \boldsymbol{\theta}(z) = \sum_{m \in \mathbb{Z}^N} \mathrm{e}^{2\pi\mathrm{i}(m,z) + \pi\mathrm{i}(m,\tau m)}, \quad z \in \mathbb{C}^N,$$

where $(\cdot,\cdot)$ denotes the—real—Euclidean inner/scalar product (for $\mathbf{A} = (A_1, A_2, \dots, A_N) \in \mathbb{E}^N$ and $\mathbf{B} = (B_1, B_2, \dots, B_N) \in \mathbb{E}^N$, $(A,B) := \sum_{k=1}^{N} A_k B_k$), with the following evenness and (quasi-) periodicity properties,

$$\boldsymbol{\theta}(-z) = \boldsymbol{\theta}(z), \qquad \boldsymbol{\theta}(z+e_j) = \boldsymbol{\theta}(z), \qquad \text{and} \qquad \boldsymbol{\theta}(z \pm \tau_j) = \mathrm{e}^{\mp 2\pi\mathrm{i} z_j - \mathrm{i}\pi\tau_{jj}} \boldsymbol{\theta}(z),$$

where e_j is the standard (basis) column vector in $\mathbb{C}^N$ with 1 in the jth entry and 0 elsewhere (see above), and $\tau_j := \tau e_j$ $(\in \mathbb{C}^N)$, $j=1,\dots,N$.

It turns out that, for the analysis of this work, the following multi-valued functions are essential:

- $(R_e(z))^{1/2} := (\prod_{k=0}^{N}(z-b_k^e)(z-a_{k+1}^e))^{1/2}$, where, with the identification $a_{N+1}^e \equiv a_0^e$ (as points on the complex sphere, $\overline{\mathbb{C}}$) and with the point at infinity lying on the (open) interval (a_0^e, b_0^e), $-\infty < a_0^e < b_0^e < a_1^e < b_1^e < \cdots < a_N^e < b_N^e < +\infty$, a_0^e $(\equiv a_{N+1}^e) \neq -\infty, 0$, and $b_N^e \neq 0, +\infty$ (see Figure 1);

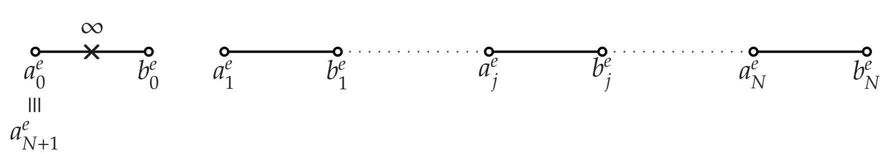

Figure 1: Union of (open) intervals in the complex z-plane

- $(R_o(z))^{1/2} := (\prod_{k=0}^{N}(z-b_k^o)(z-a_{k+1}^o))^{1/2}$, where, with the identification $a_{N+1}^o \equiv a_0^o$ (as points on the complex sphere, $\overline{\mathbb{C}}$) and with the point at infinity lying on the (open) interval (a_0^o, b_0^o), $-\infty < a_0^o < b_0^o < a_1^o < b_1^o < \cdots < a_N^o < b_N^o < +\infty$, a_0^o $(\equiv a_{N+1}^o) \neq -\infty, 0$, and $b_N^o \neq 0, +\infty$ (see Figure 2).

The functions $R_e(z)$ and $R_o(z)$, respectively, are unital polynomials $(\in \mathbb{R}[z])$ of even degree $(\deg(R_e(z)) = \deg(R_o(z)) = 2(N+1))$ whose (simple) roots/zeros are $\{b_{j-1}^e, a_j^e\}_{j=1}^{N+1}$ $(a_{N+1}^e \equiv a_0^e)$ and $\{b_{j-1}^o, a_j^o\}_{j=1}^{N+1}$ $(a_{N+1}^o \equiv a_0^o)$. The basic ingredients associated with the construction of the hyperelliptic Riemann surfaces of genus N corresponding, respectively, to the multi-valued functions $y^2 = R_e(z)$ and $y^2 = R_o(z)$ was given above. One now uses the above construction; but particularised to the cases of the polynomials $R_e(z)$ and $R_o(z)$, to arrive at the following:

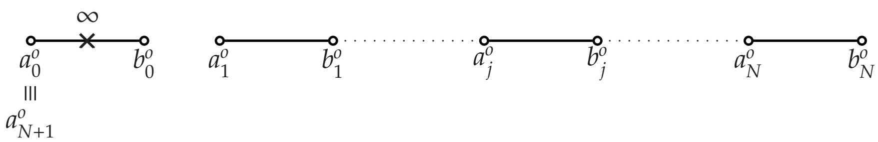

Figure 2: Union of (open) intervals in the complex z-plane

$\boxed{\sqrt{R_e(z)}}$ Let $\mathcal{Y}_e$ denote the two-sheeted Riemann surface of genus N associated with $y^2 = R_e(z)$, with $R_e(z)$ as characterised above: the first/upper (resp., second/lower) sheet of $\mathcal{Y}_e$ is denoted by $\mathcal{Y}_e^+$ (resp., $\mathcal{Y}_e^-$), points on the first/upper (resp., second/lower) sheet are represented as $z^+ := (z, +(R_e(z))^{1/2})$ (resp., $z^- := (z, -(R_e(z))^{1/2})$), where, as points on the plane $\mathbb{C}$, $z^+ = z^- = z$, and the single-valued branch for the square root of the (multi-valued) function $(R_e(z))^{1/2}$ is chosen such that $z^{-(N+1)}(R_e(z))^{1/2} \sim_{\substack{z\to\infty \\ z\in\mathcal{Y}_e^{\pm}}} \pm 1$. $\mathcal{Y}_e$ is realised as a (two-sheeted) branched/ramified covering of the Riemann sphere such that its two sheets are two identical copies of $\mathbb{C}$ with branch cuts (slits) along the intervals $(a_0^e, b_0^e), (a_1^e, b_1^e), \dots, (a_N^e, b_N^e)$ and pasted/glued together along $\cup_{j=1}^{N+1}(a_{j-1}^e, b_{j-1}^e)$ ($a_0^e \equiv a_{N+1}^e$) in such a way that the cycles $\boldsymbol{\alpha}_0^e$ and $\{\boldsymbol{\alpha}_j^e, \boldsymbol{\beta}_j^e\}$, $j = 1, \dots, N$, where the latter forms the canonical $\mathbf{1}$-homology basis for $\mathcal{Y}_e$, are characterised by the fact that (the closed contours) $\boldsymbol{\alpha}_j^e$, $j = 0, \dots, N$, lie on $\mathcal{Y}_e^+$, and (the closed contours) $\boldsymbol{\beta}_j^e$, $j = 1, \dots, N$, pass from $\mathcal{Y}_e^+$ (starting from the slit (a_j^e, b_j^e)), through the slit (a_0^e, b_0^e) to $\mathcal{Y}_e^-$, and back again to $\mathcal{Y}_e^+$ through the slit (a_j^e, b_j^e) (see Figure 3).

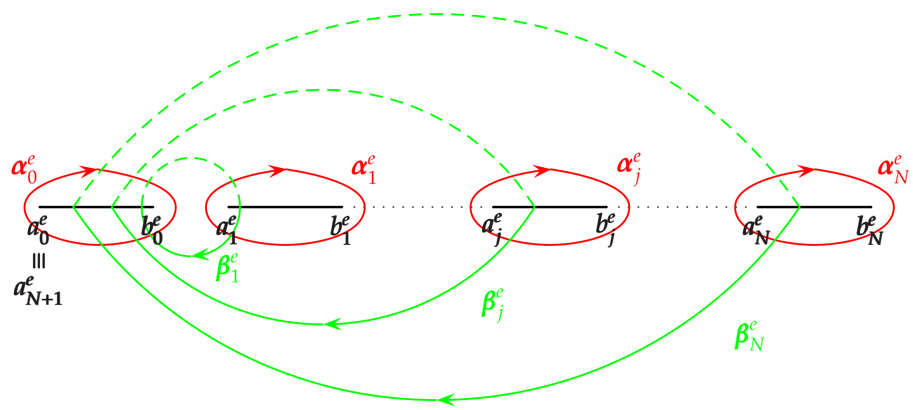

Figure 3: The Riemann surface $\mathcal{Y}_e$ of $y^2 = \prod_{k=0}^{N}(z-b_k^e)(z-a_{k+1}^e)$, $a_{N+1}^e \equiv a_0^e$. The solid (resp., dashed) lines are on the first/upper (resp., second/lower) sheet of $\mathcal{Y}_e$, denoted $\mathcal{Y}_e^+$ (resp., $\mathcal{Y}_e^-$).

The canonical $\mathbf{1}$-homology basis $\{\boldsymbol{\alpha}_j^e, \boldsymbol{\beta}_j^e\}$, $j = 1, \dots, N$, generates, on $\mathcal{Y}_e$, the (corresponding) $\boldsymbol{\alpha}^e$-normalised basis of holomorphic Abelian differentials (one-forms) $\{\omega_1^e, \omega_2^e, \dots, \omega_N^e\}$, where $\omega_j^e := \sum_{k=1}^{N} \frac{c_{jk}^e z^{N-k}}{\sqrt{R_e(z)}}\,\mathrm{d}z$, $c_{jk}^e \in \mathbb{C}$, $j = 1, \dots, N$, and $\oint_{\boldsymbol{\alpha}_k^e} \omega_j^e = \delta_{kj}$, $k, j = 1, \dots, N$: ω_l^e, $l = 1, \dots, N$, is real valued on $\cup_{j=1}^{N+1}(a_{j-1}^e, b_{j-1}^e)$, and has exactly one (real) root in any (open) interval (a_{j-1}^e, b_{j-1}^e), $j = 1, \dots, N+1$; furthermore, in the intervals (b_{j-1}^e, a_j^e), $j = 1, \dots, N$, ω_l^e, $l = 1, \dots, N$, take non-zero, pure imaginary values. Let $\boldsymbol{\omega}^e := (\omega_1^e, \omega_2^e, \dots, \omega_N^e)$ denote the basis of holomorphic one-forms on $\mathcal{Y}_e$ as normalised above with the associated $N{\times}N$ Riemann matrix of $\boldsymbol{\beta}^e$-periods, $\tau^e = (\tau^e)_{i,j=1,\dots,N} := (\oint_{\boldsymbol{\beta}_j^e} \omega_i^e)_{i,j=1,\dots,N}$: the Riemann matrix is symmetric ($\tau_{ij}^e = \tau_{ji}^e$) and pure imaginary, $-\mathrm{i}\tau^e$ is positive definite ($\mathrm{Im}(\tau_{jj}^e) > 0$), and $\det(\tau^e) \neq 0$ (non-degenerate). For the holomorphic Abelian differential (one-form) $\boldsymbol{\omega}^e$ defined above, choose a_{N+1}^e as the *base point*, and set $\boldsymbol{u}^e \colon \mathcal{Y}_e \to \mathrm{Jac}(\mathcal{Y}_e)$ $(:= \mathbb{C}^N/\{N + \tau^e M\}$, $(N, M) \in \mathbb{Z}^N \times \mathbb{Z}^N)$, $z \mapsto \boldsymbol{u}^e(z) := \int_{a_{N+1}^e}^{z} \boldsymbol{\omega}^e$, where the integration from a_{N+1}^e to z ($\in \mathcal{Y}_e$) is taken along any path on $\mathcal{Y}_e^+$.

Remark 2.1.2. From the representation $\omega_j^e = \sum_{k=1}^N \frac{c_{jk}^e z^{N-k}}{\sqrt{R_e(z)}}\,\mathrm{d}z$, $j=1,\dots,N$, and the normalisation condition $\oint_{\boldsymbol{\alpha}_k^e} \omega_j^e = \delta_{kj}$, $k,j=1,\dots,N$, one shows that c_{jk}^e, $k,j=1,\dots,N$, are obtained from

$$\begin{pmatrix} c_{11}^e & c_{12}^e & \cdots & c_{1N}^e \\ c_{21}^e & c_{22}^e & \cdots & c_{2N}^e \\ \vdots & \vdots & \ddots & \vdots \\ c_{N1}^e & c_{N2}^e & \cdots & c_{NN}^e \end{pmatrix} = \widetilde{\mathfrak{S}}_e^{-1}, \tag{E1}$$

where

$$\widetilde{\mathfrak{S}}_e := \begin{pmatrix} \oint_{\boldsymbol{\alpha}_1^e} \frac{\mathrm{d}s_1}{\sqrt{R_e(s_1)}} & \oint_{\boldsymbol{\alpha}_2^e} \frac{\mathrm{d}s_2}{\sqrt{R_e(s_2)}} & \cdots & \oint_{\boldsymbol{\alpha}_N^e} \frac{\mathrm{d}s_N}{\sqrt{R_e(s_N)}} \\ \oint_{\boldsymbol{\alpha}_1^e} \frac{s_1\mathrm{d}s_1}{\sqrt{R_e(s_1)}} & \oint_{\boldsymbol{\alpha}_2^e} \frac{s_2\mathrm{d}s_2}{\sqrt{R_e(s_2)}} & \cdots & \oint_{\boldsymbol{\alpha}_N^e} \frac{s_N\mathrm{d}s_N}{\sqrt{R_e(s_N)}} \\ \vdots & \vdots & \ddots & \vdots \\ \oint_{\boldsymbol{\alpha}_1^e} \frac{s_1^{N-1}\mathrm{d}s_1}{\sqrt{R_e(s_1)}} & \oint_{\boldsymbol{\alpha}_2^e} \frac{s_2^{N-1}\mathrm{d}s_2}{\sqrt{R_e(s_2)}} & \cdots & \oint_{\boldsymbol{\alpha}_N^e} \frac{s_N^{N-1}\mathrm{d}s_N}{\sqrt{R_e(s_N)}} \end{pmatrix}. \tag{E2}$$

For a (representation-independent) proof of the fact that $\det(\widetilde{\mathfrak{S}}_e) \neq 0$, see, for example, Chapter 10, Section 10–2, of [87]. ■

Set (see Section 4), for $z \in \mathbb{C}_+$, $\gamma^e(z) := (\prod_{k=1}^{N+1}(z-b_{k-1}^e)(z-a_k^e)^{-1})^{1/4}$, and, for $z \in \mathbb{C}_-$, $\gamma^e(z) := -\mathrm{i}(\prod_{k=1}^{N+1}(z-b_{k-1}^e)(z-a_k^e)^{-1})^{1/4}$. It is shown in Section 4 that $\gamma^e(z) \underset{z\in\mathcal{Y}_e^\pm}{=_{z\to\infty}} (-\mathrm{i})^{(1\mp1)/2} \cdot (1+\mathcal{O}(z^{-1}))$, and

$$\left\{z_j^{e,\pm}\right\}_{j=1}^N = \left\{z^\pm \in \mathcal{Y}_e^\pm;\ (\gamma^e(z) \mp (\gamma^e(z))^{-1})|_{z=z^\pm} = 0\right\},$$

with $z_j^{e,\pm} \in (a_j^e, b_j^e)^\pm$ $(\subset \mathcal{Y}_e^\pm)$, $j=1,\dots,N$, where, as points on the plane, $z_j^{e,+} = z_j^{e,-} := z_j^e$, $j=1,\dots,N$ (of course, on the plane, $z_j^e \in (a_j^e, b_j^e)$, $j=1,\dots,N$).

Corresponding to $\mathcal{Y}_e$, define $\boldsymbol{d}_e := -\boldsymbol{K}_e - \sum_{j=1}^N \int_{a_{N+1}^e}^{z_j^{e,-}} \boldsymbol{\omega}^e$ $(\in \mathbb{C}^N)$, where $\boldsymbol{K}_e$ is the associated ('even') vector of Riemann constants, and the integration from a_{N+1}^e to $z_j^{e,-}$, $j=1,\dots,N$, is taken along a fixed path in $\mathcal{Y}_e^-$. It is shown in Chapter VII of [88] that $\boldsymbol{K}_e = \sum_{j=1}^N \int_{a_j^e}^{a_{N+1}^e} \boldsymbol{\omega}^e$; furthermore, $\boldsymbol{K}_e$ is a point of order 2, that is, $2\boldsymbol{K}_e = 0$ and $s\boldsymbol{K}_e \neq 0$ for $0<s<2$. Recalling the definition of $\boldsymbol{\omega}^e$ and that $z^{-(N+1)}(R_e(z))^{1/2} \sim_{\substack{z\to\infty \\ z\in\mathbb{C}_\pm}} \pm1$, using the fact that $\boldsymbol{K}_e$ is a point of order 2, one arrives at

$$\begin{aligned} \boldsymbol{d}_e &= -\boldsymbol{K}_e - \sum_{j=1}^N \int_{a_{N+1}^e}^{z_j^{e,-}} \boldsymbol{\omega}^e = \boldsymbol{K}_e - \sum_{j=1}^N \int_{a_{N+1}^e}^{z_j^{e,-}} \boldsymbol{\omega}^e = -\boldsymbol{K}_e + \sum_{j=1}^N \int_{a_{N+1}^e}^{z_j^{e,+}} \boldsymbol{\omega}^e = \boldsymbol{K}_e + \sum_{j=1}^N \int_{a_{N+1}^e}^{z_j^{e,+}} \boldsymbol{\omega}^e \\ &= -\sum_{j=1}^N \int_{a_j^e}^{z_j^{e,-}} \boldsymbol{\omega}^e = \sum_{j=1}^N \int_{a_j^e}^{z_j^{e,+}} \boldsymbol{\omega}^e. \end{aligned}$$

Associated with the Riemann matrix of $\boldsymbol{\beta}^e$-periods, τ^e, is the ('even') Riemann theta function:

$$\boldsymbol{\theta}(z;\tau^e) =: \boldsymbol{\theta}^e(z) = \sum_{m\in\mathbb{Z}^N} \mathrm{e}^{2\pi\mathrm{i}(m,z)+\pi\mathrm{i}(m,\tau^e m)}, \quad z \in \mathbb{C}^N; \tag{2.1}$$

$\boldsymbol{\theta}^e(z)$ has the following evenness and (quasi-) periodicity properties,

$$\boldsymbol{\theta}^e(-z) = \boldsymbol{\theta}^e(z), \qquad \boldsymbol{\theta}^e(z+e_j) = \boldsymbol{\theta}^e(z), \qquad \text{and} \qquad \boldsymbol{\theta}^e(z\pm\tau_j^e) = \mathrm{e}^{\mp2\pi\mathrm{i}z_j - \mathrm{i}\pi\tau_{jj}^e}\boldsymbol{\theta}^e(z),$$

where $\tau_j^e := \tau^e e_j$ $(\in \mathbb{C}^N)$, $j=1,\dots,N$. Extensive use of this entire apparatus will be made in Section 4.

$\boxed{\sqrt{R_o(z)}}$ Let $\mathcal{Y}_o$ denote the two-sheeted Riemann surface of genus N associated with $y^2 = R_o(z)$, with $R_o(z)$ as characterised above: the first/upper (resp., second/lower) sheet of $\mathcal{Y}_o$ is denoted

by $\mathcal{Y}_o^+$ (resp., $\mathcal{Y}_o^-$), points on the first/upper (resp., second/lower) sheet are represented as $z^+ := (z, +(R_o(z))^{1/2})$ (resp., $z^- := (z, -(R_o(z))^{1/2})$), where, as points on the plane $\mathbb{C}$, $z^+ = z^- = z$, and the single-valued branch for the square root of the (multi-valued) function $(R_o(z))^{1/2}$ is chosen such that $z^{-(N+1)}(R_o(z))^{1/2} \sim_{\substack{z \to \infty \\ z \in \mathcal{Y}_o^\pm}} \pm 1$. $\mathcal{Y}_o$ is realised as a (two-sheeted) branched/ramified covering of the Riemann sphere such that its two sheets are two identical copies of $\mathbb{C}$ with branch cuts (slits) along the intervals $(a_0^o, b_0^o), (a_1^o, b_1^o), \ldots, (a_N^o, b_N^o)$ and pasted/glued together along $\cup_{j=1}^{N+1}(a_{j-1}^o, b_{j-1}^o)$ $(a_0^o \equiv a_{N+1}^o)$ in such a way that the cycles $\boldsymbol{\alpha}_0^o$ and $\{\boldsymbol{\alpha}_j^o, \boldsymbol{\beta}_j^o\}$, $j = 1, \ldots, N$, where the latter forms the canonical $\mathbf{1}$-homology basis for $\mathcal{Y}_o$, are characterised by the fact that (the closed contours) $\boldsymbol{\alpha}_j^o$, $j = 0, \ldots, N$, lie on $\mathcal{Y}_o^+$, and (the closed contours) $\boldsymbol{\beta}_j^o$, $j = 1, \ldots, N$, pass from $\mathcal{Y}_o^+$ (starting from the slit (a_j^o, b_j^o)), through the slit (a_0^o, b_0^o) to $\mathcal{Y}_o^-$, and back again to $\mathcal{Y}_o^+$ through the slit (a_j^o, b_j^o) (see Figure 4).

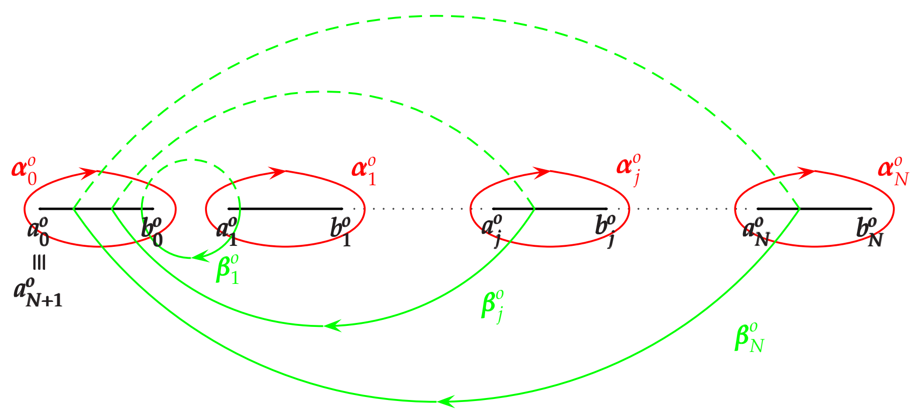

Figure 4: The Riemann surface $\mathcal{Y}_o$ of $y^2 = \prod_{k=0}^N (z - b_k^o)(z - a_{k+1}^o)$, $a_{N+1}^o \equiv a_0^o$. The solid (resp., dashed) lines are on the first/upper (resp., second/lower) sheet of $\mathcal{Y}_o$, denoted $\mathcal{Y}_o^+$ (resp., $\mathcal{Y}_o^-$).

The canonical $\mathbf{1}$-homology basis $\{\boldsymbol{\alpha}_j^o, \boldsymbol{\beta}_j^o\}$, $j = 1, \ldots, N$, generates, on $\mathcal{Y}_o$, the (corresponding) $\boldsymbol{\alpha}^o$-normalised basis of holomorphic Abelian differentials (one-forms) $\{\omega_1^o, \omega_2^o, \ldots, \omega_N^o\}$, where $\omega_j^o := \sum_{k=1}^N \frac{c_{jk}^o z^{N-k}}{\sqrt{R_o(z)}} \mathrm{d}z$, $c_{jk}^o \in \mathbb{C}$, $j = 1, \ldots, N$, and $\oint_{\boldsymbol{\alpha}_k^o} \omega_j^o = \delta_{kj}$, $k, j = 1, \ldots, N$: ω_l^o, $l = 1, \ldots, N$, is real valued on $\cup_{j=1}^{N+1}(a_{j-1}^o, b_{j-1}^o)$, and has exactly one (real) root in any (open) interval (a_{j-1}^o, b_{j-1}^o), $j = 1, \ldots, N+1$; furthermore, in the intervals (b_{j-1}^o, a_j^o), $j = 1, \ldots, N$, ω_l^o, $l = 1, \ldots, N$, take non-zero, pure imaginary values. Let $\boldsymbol{\omega}^o := (\omega_1^o, \omega_2^o, \ldots, \omega_N^o)$ denote the basis of holomorphic one-forms on $\mathcal{Y}_o$ as normalised above with the associated $N \times N$ Riemann matrix of $\boldsymbol{\beta}^o$-periods, $\tau^o = (\tau^o)_{i,j=1,\ldots,N} := (\oint_{\boldsymbol{\beta}_j^o} \omega_i^o)_{i,j=1,\ldots,N}$: the Riemann matrix is symmetric ($\tau_{ij}^o = \tau_{ji}^o$) and pure imaginary, $-\mathrm{i}\tau^o$ is positive definite ($\mathrm{Im}(\tau_{jj}^o) > 0$), and $\det(\tau^o) \neq 0$ (non-degenerate). For the holomorphic Abelian differential (one-form) $\boldsymbol{\omega}^o$ defined above, choose a_{N+1}^o as the base point, and set $\boldsymbol{u}^o : \mathcal{Y}_o \to \mathrm{Jac}(\mathcal{Y}_o)$ $(:= \mathbb{C}^N / \{N + \tau^o M\}$, $(N, M) \in \mathbb{Z}^N \times \mathbb{Z}^N)$, $z \mapsto \boldsymbol{u}^o(z) := \int_{a_{N+1}^o}^z \boldsymbol{\omega}^o$, where the integration from a_{N+1}^o to z $(\in \mathcal{Y}_o)$ is taken along any path on $\mathcal{Y}_o^+$.

Remark 2.1.3. From the representation $\omega_j^o = \sum_{k=1}^N \frac{c_{jk}^o z^{N-k}}{\sqrt{R_o(z)}} \mathrm{d}z$, $j = 1, \ldots, N$, and the normalisation condition $\oint_{\boldsymbol{\alpha}_k^o} \omega_j^o = \delta_{kj}$, $k, j = 1, \ldots, N$, one shows that c_{jk}^o, $k, j = 1, \ldots, N$, are obtained from

$$\begin{pmatrix} c_{11}^o & c_{12}^o & \cdots & c_{1N}^o \\ c_{21}^o & c_{22}^o & \cdots & c_{2N}^o \\ \vdots & \vdots & \ddots & \vdots \\ c_{N1}^o & c_{N2}^o & \cdots & c_{NN}^o \end{pmatrix} = \widetilde{\mathfrak{S}}_o^{-1}, \tag{O1}$$

where

$$\widetilde{\mathfrak{S}}_o := \begin{pmatrix} \oint_{\boldsymbol{\alpha}_1^o} \frac{\mathrm{d}s_1}{\sqrt{R_o(s_1)}} & \oint_{\boldsymbol{\alpha}_2^o} \frac{\mathrm{d}s_2}{\sqrt{R_o(s_2)}} & \cdots & \oint_{\boldsymbol{\alpha}_N^o} \frac{\mathrm{d}s_N}{\sqrt{R_o(s_N)}} \\ \oint_{\boldsymbol{\alpha}_1^o} \frac{s_1\mathrm{d}s_1}{\sqrt{R_o(s_1)}} & \oint_{\boldsymbol{\alpha}_2^o} \frac{s_2\mathrm{d}s_2}{\sqrt{R_o(s_2)}} & \cdots & \oint_{\boldsymbol{\alpha}_N^o} \frac{s_N\mathrm{d}s_N}{\sqrt{R_o(s_N)}} \\ \vdots & \vdots & \ddots & \vdots \\ \oint_{\boldsymbol{\alpha}_1^o} \frac{s_1^{N-1}\mathrm{d}s_1}{\sqrt{R_o(s_1)}} & \oint_{\boldsymbol{\alpha}_2^o} \frac{s_2^{N-1}\mathrm{d}s_2}{\sqrt{R_o(s_2)}} & \cdots & \oint_{\boldsymbol{\alpha}_N^o} \frac{s_N^{N-1}\mathrm{d}s_N}{\sqrt{R_o(s_N)}} \end{pmatrix}. \tag{O2}$$

For a (representation-independent) proof of the fact that $\det(\widetilde{\mathfrak{S}}_o) \neq 0$, see, for example, Chapter 10, Section 10–2, of [87]. ■

Set (see [51]), for $z \in \mathbb{C}_+$, $\gamma^o(z) := (\prod_{k=1}^{N+1}(z-b_{k-1}^o)(z-a_k^o)^{-1})^{1/4}$, and, for $z \in \mathbb{C}_-$, $\gamma^o(z) := -\mathrm{i}(\prod_{k=1}^{N+1}(z-b_{k-1}^o)(z-a_k^o)^{-1})^{1/4}$. It is shown in [51] that $\gamma^o(z) \underset{z\in\mathcal{Y}_o^{\pm}}{=}_{z\to 0} (-\mathrm{i})^{(1\mp1)/2}\gamma^o(0)(1+\mathcal{O}(z))$, where $\gamma^o(0) := (\prod_{k=1}^{N+1} b_{k-1}^o/a_k^o)^{1/4} > 0$, and a set of N upper-edge and lower-edge finite-length-gap roots/zeros are

$$\left\{z_j^{o,\pm}\right\}_{j=1}^N = \left\{z^{\pm} \in \mathcal{Y}_o^{\pm};\ ((\gamma^o(0))^{-1}\gamma^o(z) \mp \gamma^o(0)(\gamma^o(z))^{-1})|_{z=z^{\pm}} = 0\right\},$$

with $z_j^{o,\pm} \in (a_j^o, b_j^o)^{\pm}$ $(\subset \mathcal{Y}_o^{\pm})$, $j=1,\dots,N$, where, as points on the plane, $z_j^{o,+} = z_j^{o,-} := z_j^o$, $j=1,\dots,N$ (of course, on the plane, $z_j^o \in (a_j^o, b_j^o)$, $j=1,\dots,N$).

Corresponding to $\mathcal{Y}_o$, define $\boldsymbol{d}_o := -\boldsymbol{K}_o - \sum_{j=1}^N \int_{a_{N+1}^o}^{z_j^{o,-}} \boldsymbol{\omega}^o$ $(\in \mathbb{C}^N)$, where $\boldsymbol{K}_o$ is the associated ('odd') vector of Riemann constants, and the integration from a_{N+1}^o to $z_j^{o,-}$, $j=1,\dots,N$, is taken along a fixed path in $\mathcal{Y}_o^-$. It is shown in Chapter VII of [88] that $\boldsymbol{K}_o = \sum_{j=1}^N \int_{a_j^o}^{a_{N+1}^o} \boldsymbol{\omega}^o$; furthermore, $\boldsymbol{K}_o$ is a point of order 2. Recalling the definition of $\boldsymbol{\omega}^o$ and that $z^{-(N+1)}(R_o(z))^{1/2} \underset{z\in\mathbb{C}_\pm}{\sim}_{z\to\infty} \pm 1$, using the fact that $\boldsymbol{K}_o$ is a point of order 2, one arrives at

$$\begin{aligned} \boldsymbol{d}_o &= -\boldsymbol{K}_o - \sum_{j=1}^N \int_{a_{N+1}^o}^{z_j^{o,-}} \boldsymbol{\omega}^o = \boldsymbol{K}_o - \sum_{j=1}^N \int_{a_{N+1}^o}^{z_j^{o,-}} \boldsymbol{\omega}^o = -\boldsymbol{K}_o + \sum_{j=1}^N \int_{a_{N+1}^o}^{z_j^{o,+}} \boldsymbol{\omega}^o = \boldsymbol{K}_o + \sum_{j=1}^N \int_{a_{N+1}^o}^{z_j^{o,+}} \boldsymbol{\omega}^o \\ &= -\sum_{j=1}^N \int_{a_j^o}^{z_j^{o,-}} \boldsymbol{\omega}^o = \sum_{j=1}^N \int_{a_j^o}^{z_j^{o,+}} \boldsymbol{\omega}^o. \end{aligned}$$

Associated with the Riemann matrix of $\boldsymbol{\beta}^o$-periods, τ^o, is the ('odd') Riemann theta function:

$$\boldsymbol{\theta}(z;\tau^o) =: \boldsymbol{\theta}^o(z) = \sum_{m\in\mathbb{Z}^N} \mathrm{e}^{2\pi\mathrm{i}(m,z)+\pi\mathrm{i}(m,\tau^o m)}, \quad z \in \mathbb{C}^N;$$

$\boldsymbol{\theta}^o(z)$ has the following evenness and (quasi-) periodicity properties,

$$\boldsymbol{\theta}^o(-z) = \boldsymbol{\theta}^o(z), \qquad \boldsymbol{\theta}^o(z+e_j) = \boldsymbol{\theta}^o(z), \qquad \text{and} \qquad \boldsymbol{\theta}^o(z \pm \tau_j^o) = \mathrm{e}^{\mp 2\pi\mathrm{i}z_j - \mathrm{i}\pi\tau_{jj}^o}\boldsymbol{\theta}^o(z),$$

where $\tau_j^o := \tau^o e_j$ $(\in \mathbb{C}^N)$, $j=1,\dots,N$. This entire latter apparatus is used extensively in [51].

2.2 The Riemann-Hilbert Problems for the Monic OLPs

In this subsection, the RHPs corresponding to the even degree and odd degree monic OLPs $\boldsymbol{\pi}_{2n}(z)$ and $\boldsymbol{\pi}_{2n+1}(z)$, defined, respectively, in Equations (1.4) and (1.5), are formulated *à la* Fokas-Its-Kitaev [53, 54]. Furthermore, integral representations for the even degree and odd degree monic OLPs are also obtained.

Consider the varying exponential measure $\widetilde{\mu}$ $(\in \mathcal{M}_1(\mathbb{R}))$ given by $\mathrm{d}\widetilde{\mu}(z) = \mathrm{e}^{-\mathcal{N}V(z)}\,\mathrm{d}z$, $\mathcal{N} \in \mathbb{N}$, where (the external field) $V\colon \mathbb{R}\setminus\{0\} \to \mathbb{R}$ satisfies conditions (V1)–(V3). The RHPs which characterise the even degree and odd degree monic OLPs are now stated.

RHP1. *Let* $V\colon \mathbb{R}\setminus\{0\}\to\mathbb{R}$ *satisfy conditions* (V1)–(V3). *Find* $\overset{e}{\mathrm{Y}}\colon \mathbb{C}\setminus\mathbb{R}\to \mathrm{SL}_2(\mathbb{C})$ *solving:* (i) $\overset{e}{\mathrm{Y}}(z)$ *is holomorphic for* $z\in\mathbb{C}\setminus\mathbb{R}$; (ii) *the boundary values* $\overset{e}{\mathrm{Y}}_{\pm}(z):=\lim_{\substack{z'\to z\\ \pm\mathrm{Im}(z')>0}}\overset{e}{\mathrm{Y}}(z')$ *satisfy the jump condition*

$$\overset{e}{\mathrm{Y}}_{+}(z)=\overset{e}{\mathrm{Y}}_{-}(z)\left(\mathrm{I}+\mathrm{e}^{-\mathcal{N}V(z)}\sigma_{+}\right),\quad z\in\mathbb{R};$$

(iii) $\overset{e}{\mathrm{Y}}(z)z^{-n\sigma_3}\underset{\substack{z\to\infty\\ z\in\mathbb{C}\setminus\mathbb{R}}}{=}\mathrm{I}+\mathcal{O}(z^{-1})$; *and* (iv) $\overset{e}{\mathrm{Y}}(z)z^{n\sigma_3}\underset{\substack{z\to 0\\ z\in\mathbb{C}\setminus\mathbb{R}}}{=}\mathcal{O}(1)$.

RHP2. *Let* $V\colon \mathbb{R}\setminus\{0\}\to\mathbb{R}$ *satisfy conditions* (V1)–(V3). *Find* $\overset{o}{\mathrm{Y}}\colon \mathbb{C}\setminus\mathbb{R}\to \mathrm{SL}_2(\mathbb{C})$ *solving:* (i) $\overset{o}{\mathrm{Y}}(z)$ *is holomorphic for* $z\in\mathbb{C}\setminus\mathbb{R}$; (ii) *the boundary values* $\overset{o}{\mathrm{Y}}_{\pm}(z):=\lim_{\substack{z'\to z\\ \pm\mathrm{Im}(z')>0}}\overset{o}{\mathrm{Y}}(z')$ *satisfy the jump condition*

$$\overset{o}{\mathrm{Y}}_{+}(z)=\overset{o}{\mathrm{Y}}_{-}(z)\left(\mathrm{I}+\mathrm{e}^{-\mathcal{N}V(z)}\sigma_{+}\right),\quad z\in\mathbb{R};$$

(iii) $\overset{o}{\mathrm{Y}}(z)z^{n\sigma_3}\underset{\substack{z\to 0\\ z\in\mathbb{C}\setminus\mathbb{R}}}{=}\mathrm{I}+\mathcal{O}(z)$; *and* (iv) $\overset{o}{\mathrm{Y}}(z)z^{-(n+1)\sigma_3}\underset{\substack{z\to\infty\\ z\in\mathbb{C}\setminus\mathbb{R}}}{=}\mathcal{O}(1)$.

Lemma 2.2.1. *Let* $\overset{e}{\mathrm{Y}}\colon \mathbb{C}\setminus\mathbb{R}\to\mathrm{SL}_2(\mathbb{C})$ *solve* **RHP1**. **RHP1** *possesses a unique solution given by:* (i) *for* $n=0$,

$$\overset{e}{\mathrm{Y}}(z)=\begin{pmatrix}1 & \int_{\mathbb{R}}\frac{\exp(-\mathcal{N}V(s))}{s-z}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\\ 0 & 1\end{pmatrix},\quad z\in\mathbb{C}\setminus\mathbb{R},$$

where $\boldsymbol{\pi}_0(z):=\overset{e}{\mathrm{Y}}_{11}(z)\equiv 1$, *with* $\overset{e}{\mathrm{Y}}_{11}(z)$ *the* $(1\,1)$*-element of* $\overset{e}{\mathrm{Y}}(z)$; *and* (ii) *for* $n\in\mathbb{N}$,

$$\overset{e}{\mathrm{Y}}(z)=\begin{pmatrix}\boldsymbol{\pi}_{2n}(z) & \int_{\mathbb{R}}\frac{\boldsymbol{\pi}_{2n}(s)\exp(-\mathcal{N}V(s))}{s-z}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\\ \overset{e}{\mathrm{Y}}_{21}(z) & \int_{\mathbb{R}}\frac{\overset{e}{\mathrm{Y}}_{21}(s)\exp(-\mathcal{N}V(s))}{s-z}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\end{pmatrix},\quad z\in\mathbb{C}\setminus\mathbb{R},\tag{2.2}$$

where $\overset{e}{\mathrm{Y}}_{21}\colon\mathbb{C}^{*}\to\mathbb{C}$ *denotes the* $(2\,1)$*-element of* $\overset{e}{\mathrm{Y}}(z)$, *and* $\boldsymbol{\pi}_{2n}(z)$ *is the even degree monic* OLP *defined in Equation* (1.4).

Proof. Set $\widetilde{w}(z):=\exp(-\mathcal{N}V(z))$, $\mathcal{N}\in\mathbb{N}$, where $V\colon\mathbb{R}\setminus\{0\}\to\mathbb{R}$ satisfies conditions (V1)–(V3). Since $\int_{\mathbb{R}}s^{j}\widetilde{w}(s)\,\mathrm{d}s<\infty$, $j\in\mathbb{Z}$, it follows via a straightforward application of the Sokhotski-Plemelj formula that, for $n=0$, **RHP1** has the (unique) upper-triangular solution

$$\overset{e}{\mathrm{Y}}(z)=\begin{pmatrix}1 & \int_{\mathbb{R}}\frac{\widetilde{w}(s)}{s-z}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\\ 0 & 1\end{pmatrix},\quad z\in\mathbb{C}\setminus\mathbb{R},$$

where $\boldsymbol{\pi}_0(z):=\overset{e}{\mathrm{Y}}_{11}(z)\equiv 1$. Hereafter, $n\in\mathbb{N}$ is considered.

If $\overset{e}{\mathrm{Y}}\colon\mathbb{C}\setminus\mathbb{R}\to\mathrm{SL}_2(\mathbb{C})$ solves **RHP1**, then it follows from the jump condition (ii) of **RHP1** that, for the elements of the first column of $\overset{e}{\mathrm{Y}}(z)$,

$$\left(\overset{e}{\mathrm{Y}}_{j1}(z)\right)_{+}=\left(\overset{e}{\mathrm{Y}}_{j1}(z)\right)_{-}:=\overset{e}{\mathrm{Y}}_{j1}(z),\quad j=1,2,$$

and, for the elements of the second row,

$$\left(\overset{e}{\mathrm{Y}}_{j2}(z)\right)_{+}-\left(\overset{e}{\mathrm{Y}}_{j2}(z)\right)_{-}=\overset{e}{\mathrm{Y}}_{j1}(z)\widetilde{w}(z),\quad j=1,2.$$

From condition (i), the normalisation condition (iii), and the boundedness condition (iv) of **RHP1**, in particular, $\overset{e}{\mathrm{Y}}_{11}(z)z^{-n}\underset{\substack{z\to\infty\\ z\in\mathbb{C}\setminus\mathbb{R}}}{=}1+\mathcal{O}(z^{-1})$, $\overset{e}{\mathrm{Y}}_{11}(z)z^{n}\underset{\substack{z\to 0\\ z\in\mathbb{C}\setminus\mathbb{R}}}{=}\mathcal{O}(1)$, $\overset{e}{\mathrm{Y}}_{21}(z)z^{-n}\underset{\substack{z\to\infty\\ z\in\mathbb{C}\setminus\mathbb{R}}}{=}\mathcal{O}(z^{-1})$, and $\overset{e}{\mathrm{Y}}_{21}(z)z^{n}\underset{\substack{z\to 0\\ z\in\mathbb{C}\setminus\mathbb{R}}}{=}\mathcal{O}(1)$, and the fact that $\overset{e}{\mathrm{Y}}_{11}(z)$ and $\overset{e}{\mathrm{Y}}_{21}(z)$ have no jumps throughout the z-plane, it follows that $\overset{e}{\mathrm{Y}}_{11}(z)$ is a monic rational function with a pole at the origin and at the point at infinity, with representation $\overset{e}{\mathrm{Y}}_{11}(z)=\sum_{l=-n}^{n}\nu_{l}z^{l}$, where $\nu_n=1$, and $\overset{e}{\mathrm{Y}}_{21}(z)$ is a rational function with a pole at the origin and at the point

at infinity, with representation $\overset{e}{\mathrm{Y}}_{21}(z)=\sum_{l=-n}^{n-1}\nu_{l}^{\sharp}z^{l}$. Application of the Sokhotski-Plemelj formula to the jump relations for $\overset{e}{\mathrm{Y}}_{j2}(z)$, $j=1,2$, gives rise to the following Cauchy-type integral representations:

$$\overset{e}{\mathrm{Y}}_{j2}(z)=\int_{\mathbb{R}}\frac{\overset{e}{\mathrm{Y}}_{j1}(s)\widetilde{w}(s)}{s-z}\frac{\mathrm{d}s}{2\pi\mathrm{i}},\quad j=1,2,\quad z\in\mathbb{C}\setminus\mathbb{R}. \tag{CA1}$$

One now studies $\overset{e}{\mathrm{Y}}_{j1}(z)$, $j=1,2$, in more detail. From the normalisation condition (iii) of **RHP1**, in particular, $\overset{e}{\mathrm{Y}}_{12}(z)z^{n}\underset{z\in\mathbb{C}\setminus\mathbb{R}}{=_{z\to\infty}}\mathcal{O}(z^{-1})$ and $\overset{e}{\mathrm{Y}}_{22}(z)z^{n}\underset{z\in\mathbb{C}\setminus\mathbb{R}}{=_{z\to\infty}}1+\mathcal{O}(z^{-1})$, the formulae (CA1), the fact that $\int_{\mathbb{R}}s^{j}\widetilde{w}(s)\,\mathrm{d}s<\infty$, $j\in\mathbb{Z}$, and the expansion (for $|s/z|\ll 1$) $\frac{1}{s-z}=-\sum_{k=0}^{l}\frac{s^{k}}{z^{k+1}}+\frac{s^{l+1}}{z^{l+1}(s-z)}$, $l\in\mathbb{Z}_{0}^{+}$, it follows that

$$\int_{\mathbb{R}}\overset{e}{\mathrm{Y}}_{11}(s)s^{k}\widetilde{w}(s)\,\mathrm{d}s=0,\quad k=0,1,\dots,n-1,\qquad\text{and}\qquad\int_{\mathbb{R}}\overset{e}{\mathrm{Y}}_{11}(s)s^{n}\widetilde{w}(s)\,\mathrm{d}s=-2\pi\mathrm{i}\mathfrak{p}^{e},$$

for some (pure imaginary) $\mathfrak{p}^{e}$ of the form $\mathfrak{p}^{e}=\mathrm{i}\mathfrak{q}^{e}$, with $\mathfrak{q}^{e}>0$ (see below), and

$$\int_{\mathbb{R}}\overset{e}{\mathrm{Y}}_{21}(s)s^{j}\widetilde{w}(s)\,\mathrm{d}s=0,\quad j=0,1,\dots,n-2,\qquad\text{and}\qquad\int_{\mathbb{R}}\overset{e}{\mathrm{Y}}_{21}(s)s^{n-1}\widetilde{w}(s)\,\mathrm{d}s=-2\pi\mathrm{i};$$

and, from the boundedness condition (iv) of **RHP1**, in particular, $\overset{e}{\mathrm{Y}}_{12}(z)z^{-n}\underset{z\in\mathbb{C}\setminus\mathbb{R}}{=_{z\to 0}}\mathcal{O}(1)$ and $\overset{e}{\mathrm{Y}}_{22}(z)\cdot z^{-n}\underset{z\in\mathbb{C}\setminus\mathbb{R}}{=_{z\to 0}}\mathcal{O}(1)$, the formulae (CA1), the fact that $\int_{\mathbb{R}}s^{j}\widetilde{w}(s)\,\mathrm{d}s<\infty$, $j\in\mathbb{Z}$, and the expansion (for $|z/s|\ll 1$) $\frac{1}{z-s}=-\sum_{k=0}^{l}\frac{z^{k}}{s^{k+1}}+\frac{z^{l+1}}{s^{l+1}(z-s)}$, $l\in\mathbb{Z}_{0}^{+}$, it follows that

$$\int_{\mathbb{R}}\overset{e}{\mathrm{Y}}_{11}(s)s^{-k}\widetilde{w}(s)\,\mathrm{d}s=0,\quad k=1,2,\dots,n,\qquad\text{and}\qquad\int_{\mathbb{R}}\overset{e}{\mathrm{Y}}_{21}(s)s^{-j}\widetilde{w}(s)\,\mathrm{d}s=0,\quad j=1,2,\dots,n:$$

these give rise to $2n+1$ conditions for $\overset{e}{\mathrm{Y}}_{11}(z)$, and $2n$ conditions for $\overset{e}{\mathrm{Y}}_{21}(z)$. Consider, first, the $2n$ conditions for $\overset{e}{\mathrm{Y}}_{21}(z)$. Recalling that the strong moments are defined by $c_{j}:=\int_{\mathbb{R}}s^{j}\widetilde{w}(s)\,\mathrm{d}s$, $j\in\mathbb{Z}$, it follows from the representation (established above) $\overset{e}{\mathrm{Y}}_{21}(z)=\sum_{l=-n}^{n-1}\nu_{l}^{\sharp}z^{l}$ and the $2n$ conditions for $\overset{e}{\mathrm{Y}}_{21}(z)$ that

$$\sum_{l=-n}^{n-1}\nu_{l}^{\sharp}c_{l+k}=0,\quad k=-n,-(n-1),\dots,n-2,\qquad\text{and}\qquad\sum_{l=-n}^{n-1}\nu_{l}^{\sharp}c_{l+n-1}=-2\pi\mathrm{i},$$

that is,

$$\begin{pmatrix} c_{-2n} & c_{-2n+1} & \cdots & c_{-2} & c_{-1}\\ c_{-2n+1} & c_{-2n+2} & \cdots & c_{-1} & c_{0}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ c_{-2} & c_{-1} & \cdots & c_{2n-4} & c_{2n-3}\\ c_{-1} & c_{0} & \cdots & c_{2n-3} & c_{2n-2}\end{pmatrix}\begin{pmatrix}\nu_{-n}^{\sharp}\\ \nu_{-n+1}^{\sharp}\\ \vdots\\ \nu_{n-2}^{\sharp}\\ \nu_{n-1}^{\sharp}\end{pmatrix}=\begin{pmatrix}0\\ 0\\ \vdots\\ 0\\ -2\pi\mathrm{i}\end{pmatrix}.$$

This linear system of $2n$ equations for the $2n$ unknowns $\nu_{l}^{\sharp}$, $l=-n,-(n-1),\dots,n-1$, admits a unique solution if, and only if, the determinant of the coefficient matrix, in this case $H_{2n}^{(-2n)}$ (cf. Equations (1.1)), is non-zero; in fact, it will be shown that $H_{2n}^{(-2n)}>0$. An integral representation for the Hankel determinants $H_{k}^{(m)}$, $(m,k)\in\mathbb{Z}\times\mathbb{N}$, is now obtained; then the substitutions $m=-2n$ and $k=2n$ are made. In the calculations that follow, $\mathfrak{S}_{k}$ denotes the $k!$ permutations $\boldsymbol{\sigma}$ of $\{1,2,\dots,k\}$. Recalling that $c_{j}:=\int_{\mathbb{R}}s^{j}\,\mathrm{d}\widetilde{\mu}(s)$, $j\in\mathbb{Z}$, where $\mathrm{d}\widetilde{\mu}(z)=\widetilde{w}(z)\,\mathrm{d}z=\exp(-\mathcal{N}\,V(z))\,\mathrm{d}z$, and using the multi-linearity property of the determinant, via Equations (1.1), one proceeds thus (recall that $H_{0}^{(m)}:=1$):

$$H_{k}^{(m)}:=\begin{vmatrix} c_{m} & c_{m+1} & \cdots & c_{m+k-1}\\ c_{m+1} & c_{m+2} & \cdots & c_{m+k}\\ \vdots & \vdots & \ddots & \vdots\\ c_{m+k-2} & c_{m+k-1} & \cdots & c_{m+2k-3}\\ c_{m+k-1} & c_{m+k} & \cdots & c_{m+2k-2}\end{vmatrix}$$

$$
\begin{aligned}
&= \begin{vmatrix}
\int_{\mathbb{R}} s_1^{m}\,\mathrm{d}\widetilde{\mu}(s_1) & \int_{\mathbb{R}} s_2^{m+1}\,\mathrm{d}\widetilde{\mu}(s_2) & \cdots & \int_{\mathbb{R}} s_k^{m+k-1}\,\mathrm{d}\widetilde{\mu}(s_k) \\
\int_{\mathbb{R}} s_1^{m+1}\,\mathrm{d}\widetilde{\mu}(s_1) & \int_{\mathbb{R}} s_2^{m+2}\,\mathrm{d}\widetilde{\mu}(s_2) & \cdots & \int_{\mathbb{R}} s_k^{m+k}\,\mathrm{d}\widetilde{\mu}(s_k) \\
\vdots & \vdots & \ddots & \vdots \\
\int_{\mathbb{R}} s_1^{m+k-2}\,\mathrm{d}\widetilde{\mu}(s_1) & \int_{\mathbb{R}} s_2^{m+k-1}\,\mathrm{d}\widetilde{\mu}(s_2) & \cdots & \int_{\mathbb{R}} s_k^{m+2k-3}\,\mathrm{d}\widetilde{\mu}(s_k) \\
\int_{\mathbb{R}} s_1^{m+k-1}\,\mathrm{d}\widetilde{\mu}(s_1) & \int_{\mathbb{R}} s_2^{m+k}\,\mathrm{d}\widetilde{\mu}(s_2) & \cdots & \int_{\mathbb{R}} s_k^{m+2k-2}\,\mathrm{d}\widetilde{\mu}(s_k)
\end{vmatrix} \\
&= \underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}}_{k} \mathrm{d}\widetilde{\mu}(s_1)\,\mathrm{d}\widetilde{\mu}(s_2)\,\cdots\,\mathrm{d}\widetilde{\mu}(s_k)
\begin{vmatrix}
s_1^{m} & s_2^{m+1} & \cdots & s_k^{m+k-1} \\
s_1^{m+1} & s_2^{m+2} & \cdots & s_k^{m+k} \\
\vdots & \vdots & \ddots & \vdots \\
s_1^{m+k-2} & s_2^{m+k-1} & \cdots & s_k^{m+2k-3} \\
s_1^{m+k-1} & s_2^{m+k} & \cdots & s_k^{m+2k-2}
\end{vmatrix} \\
&= \underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}}_{k} \mathrm{d}\widetilde{\mu}(s_1)\,\mathrm{d}\widetilde{\mu}(s_2)\,\cdots\,\mathrm{d}\widetilde{\mu}(s_k)\, s_1^{m}s_2^{m+1}\cdots s_k^{m+k-1}
\underbrace{\begin{vmatrix}
1 & 1 & \cdots & 1 \\
s_1 & s_2 & \cdots & s_k \\
\vdots & \vdots & \ddots & \vdots \\
s_1^{k-2} & s_2^{k-2} & \cdots & s_k^{k-2} \\
s_1^{k-1} & s_2^{k-1} & \cdots & s_k^{k-1}
\end{vmatrix}}_{=:\,\mathrm{V}(s_1,s_2,\dots,s_k)} \\
&= \frac{1}{k!}\sum_{\boldsymbol{\sigma}\in\mathfrak{S}_k} \underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}}_{k} \mathrm{d}\widetilde{\mu}(s_{\sigma(1)})\,\mathrm{d}\widetilde{\mu}(s_{\sigma(2)})\,\cdots\,\mathrm{d}\widetilde{\mu}(s_{\sigma(k)}) \prod_{j=1}^{k} s_{\sigma(j)}^{m}s_{\sigma(j)}^{j-1}\,\mathrm{V}\Big(s_{\sigma(1)},s_{\sigma(2)},\dots,s_{\sigma(k)}\Big) \\
&= \frac{1}{k!}\sum_{\boldsymbol{\sigma}\in\mathfrak{S}_k} \underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}}_{k} \mathrm{d}\widetilde{\mu}(s_1)\,\mathrm{d}\widetilde{\mu}(s_2)\,\cdots\,\mathrm{d}\widetilde{\mu}(s_k)\, s_1^{m}s_2^{m}\cdots s_k^{m}\left(\mathrm{sgn}(\boldsymbol{\sigma})\prod_{j=1}^{k} s_{\sigma(j)}^{j-1}\right) \\
&\times \mathrm{V}(s_1,s_2,\dots,s_k) \\
&= \frac{1}{k!}\underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}}_{k} \mathrm{d}\widetilde{\mu}(s_1)\,\mathrm{d}\widetilde{\mu}(s_2)\,\cdots\,\mathrm{d}\widetilde{\mu}(s_k)\, s_1^{m}s_2^{m}\cdots s_k^{m}\,\mathrm{V}(s_1,s_2,\dots,s_k) \\
&\times \underbrace{\sum_{\boldsymbol{\sigma}\in\mathfrak{S}_k}\mathrm{sgn}(\boldsymbol{\sigma})s_{\sigma(1)}^{0}s_{\sigma(2)}^{1}\cdots s_{\sigma(k)}^{k-1}}_{=\,\mathrm{V}(s_1,s_2,\dots,s_k)} \quad \Rightarrow \\
H_k^{(m)} &= \frac{1}{k!}\underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}}_{k} \mathrm{d}\widetilde{\mu}(s_1)\,\mathrm{d}\widetilde{\mu}(s_2)\,\cdots\,\mathrm{d}\widetilde{\mu}(s_k)\, s_1^{m}s_2^{m}\cdots s_k^{m}(\mathrm{V}(s_1,s_2,\dots,s_k))^2\,;
\end{aligned}
$$

using the well-known determinantal formula $\mathrm{V}(s_1,s_2,\dots,s_k)=\prod_{\substack{i,j=1\\ j<i}}^{k}(s_i-s_j)$, one arrives at

$$
H_k^{(m)}=\frac{1}{k!}\underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}}_{k} s_1^{m}s_2^{m}\cdots s_k^{m}\prod_{\substack{i,l=1\\ l<i}}^{k}(s_i-s_l)^2\,\mathrm{d}\widetilde{\mu}(s_1)\,\mathrm{d}\widetilde{\mu}(s_2)\,\cdots\,\mathrm{d}\widetilde{\mu}(s_k), \quad (m,k)\in\mathbb{Z}\times\mathbb{N}. \tag{HA1}
$$

Letting $m=-2n$ and $k=2n$, it follows from the formula (HA1) that

$$
H_{2n}^{(-2n)}=\frac{1}{(2n)!}\underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}}_{2n} s_1^{-2n}s_2^{-2n}\cdots s_{2n}^{-2n}\prod_{\substack{i,l=1\\ l<i}}^{2n}(s_i-s_l)^2\,\mathrm{d}\widetilde{\mu}(s_1)\,\mathrm{d}\widetilde{\mu}(s_2)\,\cdots\,\mathrm{d}\widetilde{\mu}(s_{2n})>0,
$$

whence the existence (and uniqueness) of $\overset{e}{\mathrm{Y}}_{21}(z)$.

Similarly, it follows, from the representation (established above) $\overset{e}{\mathrm{Y}}_{11}(z)=\sum_{l=-n}^{n}\nu_l z^l$, with $\nu_n=1$, and the $2n+1$ conditions for $\overset{e}{\mathrm{Y}}_{11}(z)$, that

$$\sum_{l=-n}^{n}\nu_l c_{l+k}=0, \quad k=-n,-(n-1),\dots,n-1, \qquad \text{and} \qquad \sum_{l=-n}^{n}\nu_l c_{l+n}=-2\pi\mathrm{i}\mathfrak{p}^e,$$

that is,

$$\begin{pmatrix} c_{-2n} & \cdots & c_{-n} & \cdots & c_{-1} & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ c_{-n} & \cdots & c_0 & \cdots & c_{n-1} & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ c_{-1} & \cdots & c_{n-1} & \cdots & c_{2(n-1)} & 0 \\ c_0 & \cdots & c_n & \cdots & c_{2n-1} & 2\pi\mathrm{i} \end{pmatrix}\begin{pmatrix} \nu_{-n} \\ \vdots \\ \nu_0 \\ \vdots \\ \nu_{n-1} \\ \mathfrak{p}^e \end{pmatrix}=\begin{pmatrix} -c_0 \\ \vdots \\ -c_n \\ \vdots \\ -c_{2n-1} \\ -c_{2n} \end{pmatrix}.$$

This linear system of $2n+1$ equations for the $2n+1$ unknowns ν_l, $l=-n,-(n-1),\cdots,n-1$, and $\mathfrak{p}^e$ admits a unique solution if, and only if, the determinant of the coefficient matrix, in this case $2\pi\mathrm{i}H_{2n}^{(-2n)}$, is non-zero; but, it was shown above that $H_{2n}^{(-2n)}>0$. Furthermore, via Cramer's Rule:

$$\mathfrak{p}^e=\frac{\begin{vmatrix} c_{-2n} & \cdots & c_{-n} & \cdots & c_{-1} & -c_0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ c_{-n} & \cdots & c_0 & \cdots & c_{n-1} & -c_n \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ c_{-1} & \cdots & c_{n-1} & \cdots & c_{2(n-1)} & -c_{2n-1} \\ c_0 & \cdots & c_n & \cdots & c_{2n-1} & -c_{2n} \end{vmatrix}}{2\pi\mathrm{i}H_{2n}^{(-2n)}}=-\frac{1}{2\pi\mathrm{i}}\frac{H_{2n+1}^{(-2n)}}{H_{2n}^{(-2n)}}.$$

Using the Hankel determinant formula (HA1) with the substitutions $m=-2n$ and $k=2n+1$, one arrives at

$$H_{2n+1}^{(-2n)}=\frac{1}{(2n+1)!}\underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}}_{2n+1} s_1^{-2n}s_2^{-2n}\cdots s_{2n+1}^{-2n}\prod_{\substack{i,l=1\\ l<i}}^{2n+1}(s_i-s_l)^2\,\mathrm{d}\widetilde{\mu}(s_1)\,\mathrm{d}\widetilde{\mu}(s_2)\,\cdots\,\mathrm{d}\widetilde{\mu}(s_{2n+1})>0;$$

hence, $H_{2n+1}^{(-2n)}/H_{2n}^{(-2n)}>0$. Using, now, the fact that $\int_{\mathbb{R}}\overset{e}{\mathrm{Y}}_{11}(s)s^k\widetilde{w}(s)\,\mathrm{d}s=0$, $k=-n,-(n-1),\dots,n-1$, and the relation $\int_{\mathbb{R}}\overset{e}{\mathrm{Y}}_{11}(s)s^n\widetilde{w}(s)\,\mathrm{d}s=-2\pi\mathrm{i}\mathfrak{p}^e$, one notes, via the above formula for $\mathfrak{p}^e$, that

$$\begin{aligned}\int_{\mathbb{R}}\overset{e}{\mathrm{Y}}_{11}(s)s^n\widetilde{w}(s)\,\mathrm{d}s &= \int_{\mathbb{R}}\overset{e}{\mathrm{Y}}_{11}(s)\underbrace{\left(s^n+\nu_{n-1}s^{n-1}+\cdots+\nu_{-n}s^{-n}\right)}_{=\overset{e}{\mathrm{Y}}_{11}(s)}\widetilde{w}(s)\,\mathrm{d}s=\int_{\mathbb{R}}\overset{e}{\mathrm{Y}}_{11}(s)\overset{e}{\mathrm{Y}}_{11}(s)\widetilde{w}(s)\,\mathrm{d}s \\ &= -2\pi\mathrm{i}\mathfrak{p}^e=H_{2n+1}^{(-2n)}/H_{2n}^{(-2n)} \quad (>0);\end{aligned}$$

but the right-hand side of the latter expression (cf. Equations (1.8)) is equal to $(\xi_n^{(2n)})^{-2}=\|\overset{e}{\mathrm{Y}}_{11}(\cdot)\|_{\mathcal{L}}^2$ (>0): the existence and uniqueness of $\overset{e}{\mathrm{Y}}_{11}(z)=:\boldsymbol{\pi}_{2n}(z)$, the even degree monic OLP with respect to the inner product $\langle\cdot,\cdot\rangle_{\mathcal{L}}$, is thus established. □

Lemma 2.2.2. *Let* $\overset{o}{\mathrm{Y}}\colon\mathbb{C}\setminus\mathbb{R}\to\mathrm{SL}_2(\mathbb{C})$ *solve* **RHP2**. **RHP2** *possesses a unique solution given by:* (i) *for* $n=0$,

$$\overset{o}{\mathrm{Y}}(z)=\begin{pmatrix} z\boldsymbol{\pi}_1(z) & z\int_{\mathbb{R}}\frac{(s\boldsymbol{\pi}_1(s))\exp(-\mathcal{N}V(s))}{s(s-z)}\frac{\mathrm{d}s}{2\pi\mathrm{i}} \\ 2\pi\mathrm{i}z & 1+z\int_{\mathbb{R}}\frac{\exp(-\mathcal{N}V(s))}{s-z}\,\mathrm{d}s \end{pmatrix}, \quad z\in\mathbb{C}\setminus\mathbb{R},$$

where $\boldsymbol{\pi}_1(z)=\frac{1}{z}+\frac{\xi_0^{(1)}}{\xi_{-1}^{(1)}}$, *with* $\frac{\xi_0^{(1)}}{\xi_{-1}^{(1)}}=-\int_{\mathbb{R}}s^{-1}\exp(-\mathcal{N}V(s))\,\mathrm{d}s$, $\mathcal{N}\in\mathbb{N}$; *and* (ii) *for* $n\in\mathbb{N}$,

$$\overset{o}{\mathrm{Y}}(z)=\begin{pmatrix} z\boldsymbol{\pi}_{2n+1}(z) & z\int_{\mathbb{R}}\frac{(s\boldsymbol{\pi}_{2n+1}(s))\exp(-\mathcal{N}V(s))}{s(s-z)}\frac{\mathrm{d}s}{2\pi\mathrm{i}} \\ \overset{o}{\mathrm{Y}}_{21}(z) & z\int_{\mathbb{R}}\frac{\overset{o}{\mathrm{Y}}_{21}(s)\exp(-\mathcal{N}V(s))}{s(s-z)}\frac{\mathrm{d}s}{2\pi\mathrm{i}} \end{pmatrix}, \quad z\in\mathbb{C}\setminus\mathbb{R},$$

where $\overset{o}{\mathrm{Y}}_{21} \colon \mathbb{C}^{*} \to \mathbb{C}$ *denotes the* $(2\,1)$*-element of* $\overset{o}{\mathrm{Y}}(z)$*, and* $\boldsymbol{\pi}_{2n+1}(z)$ *is the odd degree monic* OLP *defined in Equation* (1.5).

Proof. See [51], the proof of Lemma 2.2.2. □

Corollary 2.2.1. *Let* $V \colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ *satisfy conditions* (V1)–(V3). *Let* $\boldsymbol{\pi}_{2n}(z)$ *and* $\boldsymbol{\pi}_{2n+1}(z)$ *be the even degree and odd degree monic* OLPs *with respect to the inner product* $\langle \cdot,\cdot \rangle_{\mathcal{L}}$ *defined, respectively, in Equations* (1.4) *and* (1.5), *and let* $\xi_{n}^{(2n)}$ *and* $\xi_{-n-1}^{(2n+1)}$ *be the corresponding 'even' and 'odd' norming constants, respectively. Then,* $\xi_{n}^{(2n)}$ *and* $\xi_{-n-1}^{(2n+1)}$ *have the following representations:*

$$\frac{\xi_{n}^{(2n)}}{\sqrt{2n+1}} = \sqrt{\frac{\underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}}_{2n} s_{1}^{-2n}s_{2}^{-2n}\cdots s_{2n}^{-2n}\prod_{\substack{i,l=1\\l<i}}^{2n}(s_{i}-s_{l})^{2}\,\mathrm{d}\widetilde{\mu}(s_{1})\,\mathrm{d}\widetilde{\mu}(s_{2})\cdots\mathrm{d}\widetilde{\mu}(s_{2n})}{\underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}}_{2n+1} \lambda_{1}^{-2n}\lambda_{2}^{-2n}\cdots\lambda_{2n+1}^{-2n}\prod_{\substack{i,l=1\\l<i}}^{2n+1}(\lambda_{i}-\lambda_{l})^{2}\,\mathrm{d}\widetilde{\mu}(\lambda_{1})\,\mathrm{d}\widetilde{\mu}(\lambda_{2})\cdots\mathrm{d}\widetilde{\mu}(\lambda_{2n+1})}},$$

$$\frac{\xi_{-n-1}^{(2n+1)}}{\sqrt{2(n+1)}} = \sqrt{\frac{\underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}}_{2n+1} \varpi_{1}^{-2n}\varpi_{2}^{-2n}\cdots\varpi_{2n+1}^{-2n}\prod_{\substack{i,l=1\\l<i}}^{2n+1}(\varpi_{i}-\varpi_{l})^{2}\,\mathrm{d}\widetilde{\mu}(\varpi_{1})\,\mathrm{d}\widetilde{\mu}(\varpi_{2})\cdots\mathrm{d}\widetilde{\mu}(\varpi_{2n+1})}{\underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}}_{2n+2} \varsigma_{1}^{-2n-2}\varsigma_{2}^{-2n-2}\cdots\varsigma_{2n+2}^{-2n-2}\prod_{\substack{i,l=1\\l<i}}^{2n+2}(\varsigma_{i}-\varsigma_{l})^{2}\,\mathrm{d}\widetilde{\mu}(\varsigma_{1})\,\mathrm{d}\widetilde{\mu}(\varsigma_{2})\cdots\mathrm{d}\widetilde{\mu}(\varsigma_{2n+2})}},$$

where $\mathrm{d}\widetilde{\mu}(z) := \exp(-\mathscr{N}\,V(z))\,\mathrm{d}z$, $\mathscr{N} \in \mathbb{N}$.

Proof. Consider, without loss of generality, the representation for $\xi_{n}^{(2n)}$. Recall that (cf. Equations (1.8)) $(\xi_{n}^{(2n)})^{2} = H_{2n}^{(-2n)}/H_{2n+1}^{(-2n)}$ (>0): using the integral representations for $H_{2n}^{(-2n)}$ and $H_{2n+1}^{(-2n)}$ derived in (the course of) the proof of Lemma 2.2.1, and taking positive square roots of both sides of the resulting equality, one arrives at the representation for $\xi_{n}^{(2n)}$. See [51], Corollary 2.2.1, for the proof of the representation for $\xi_{-n-1}^{(2n+1)}$. □

Proposition 2.2.1. *Let* $V \colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ *satisfy conditions* (V1)–(V3). *Let* $\boldsymbol{\pi}_{2n}(z)$ *and* $\boldsymbol{\pi}_{2n+1}(z)$ *be the even degree and odd degree monic* OLPs *with respect to the inner product* $\langle \cdot,\cdot \rangle_{\mathcal{L}}$ *defined, respectively, in Equations* (1.4) *and* (1.5). *Then,* $\boldsymbol{\pi}_{2n}(z)$ *and* $\boldsymbol{\pi}_{2n+1}(z)$ *have, respectively, the following integral representations:*

$$\begin{aligned}\boldsymbol{\pi}_{2n}(z) = \frac{z^{-n}}{(2n)!H_{2n}^{(-2n)}} \underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}}_{2n} s_{0}^{-2n}s_{1}^{-2n}\cdots s_{2n-1}^{-2n}\prod_{\substack{i,l=0\\l<i}}^{2n-1}(s_{i}-s_{l})^{2}\prod_{j=0}^{2n-1}(z-s_{j})\\ \times\mathrm{d}\widetilde{\mu}(s_{0})\,\mathrm{d}\widetilde{\mu}(s_{1})\cdots\mathrm{d}\widetilde{\mu}(s_{2n-1}),\end{aligned}$$

$$\begin{aligned}\boldsymbol{\pi}_{2n+1}(z) = -\frac{z^{-n-1}}{(2n+1)!H_{2n+1}^{(-2n)}} \underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}}_{2n+1} s_{0}^{-2n-1}s_{1}^{-2n-1}\cdots s_{2n}^{-2n-1}\prod_{\substack{i,l=0\\l<i}}^{2n}(s_{i}-s_{l})^{2}\prod_{j=0}^{2n}(z-s_{j})\\ \times\mathrm{d}\widetilde{\mu}(s_{0})\,\mathrm{d}\widetilde{\mu}(s_{1})\cdots\mathrm{d}\widetilde{\mu}(s_{2n}),\end{aligned}$$

where

$$H_{2n}^{(-2n)} = \frac{1}{(2n)!}\underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}}_{2n} \lambda_{1}^{-2n}\lambda_{2}^{-2n}\cdots\lambda_{2n}^{-2n}\prod_{\substack{i,l=1\\l<i}}^{2n}(\lambda_{i}-\lambda_{l})^{2}\,\mathrm{d}\widetilde{\mu}(\lambda_{1})\,\mathrm{d}\widetilde{\mu}(\lambda_{2})\cdots\mathrm{d}\widetilde{\mu}(\lambda_{2n}),$$

$$H_{2n+1}^{(-2n)} = \frac{1}{(2n+1)!} \underbrace{\int_{\mathbb{R}} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}}}_{2n+1} \lambda_1^{-2n} \lambda_2^{-2n} \cdots \lambda_{2n+1}^{-2n} \prod_{\substack{i,l=1 \\ l<i}}^{2n+1} (\lambda_i - \lambda_l)^2 \, \mathrm{d}\widetilde{\mu}(\lambda_1) \, \mathrm{d}\widetilde{\mu}(\lambda_2) \cdots \mathrm{d}\widetilde{\mu}(\lambda_{2n+1}),$$

with $\mathrm{d}\widetilde{\mu}(z) := \exp(-\mathcal{N} V(z)) \, \mathrm{d}z$, $\mathcal{N} \in \mathbb{N}$.

Proof. Consider, without loss of generality, the integral representation for the even degree monic OLP $\boldsymbol{\pi}_{2n}(z)$. Let $\mathfrak{S}_k$ denote the $k!$ permutations $\boldsymbol{\sigma}$ of $\{0, 1, \ldots, k-1\}$. Recalling that $c_j := \int_{\mathbb{R}} s^j \, \mathrm{d}\widetilde{\mu}(s)$, $j \in \mathbb{Z}$, where $\mathrm{d}\widetilde{\mu}(z) = \widetilde{w}(z) \, \mathrm{d}z = \exp(-\mathcal{N} V(z)) \, \mathrm{d}z$, $\mathcal{N} \in \mathbb{N}$, with $V \colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ satisfying conditons (V1)–(V3), and using the multi-linearity property of the determinant, via the determinantal representation for $\boldsymbol{\pi}_{2n}(z)$ given in Equation (1.6), one proceeds thus:

$$\begin{aligned}
\boldsymbol{\pi}_{2n}(z) &= \frac{1}{H_{2n}^{(-2n)}} \begin{vmatrix} c_{-2n} & c_{-2n+1} & \cdots & c_{-1} & z^{-n} \\ c_{-2n+1} & c_{-2n+2} & \cdots & c_0 & z^{-n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{-1} & c_0 & \cdots & c_{2n-2} & z^{n-1} \\ c_0 & c_1 & \cdots & c_{2n-1} & z^n \end{vmatrix} \\
&= \frac{z^{-n}}{H_{2n}^{(-2n)}} \begin{vmatrix} c_{-2n} & c_{-2n+1} & \cdots & c_{-1} & c_0 \\ c_{-2n+1} & c_{-2n+2} & \cdots & c_0 & c_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{-1} & c_0 & \cdots & c_{2n-2} & c_{2n-1} \\ z^0 & z^1 & \cdots & z^{2n-1} & z^{2n} \end{vmatrix} \\
&= \frac{z^{-n}}{H_{2n}^{(-2n)}} \begin{vmatrix} \int_{\mathbb{R}} s_0^{-2n} \, \mathrm{d}\widetilde{\mu}(s_0) & \int_{\mathbb{R}} s_0^{-2n+1} \, \mathrm{d}\widetilde{\mu}(s_0) & \cdots & \int_{\mathbb{R}} s_0^{0} \, \mathrm{d}\widetilde{\mu}(s_0) \\ \int_{\mathbb{R}} s_1^{-2n+1} \, \mathrm{d}\widetilde{\mu}(s_1) & \int_{\mathbb{R}} s_1^{-2n+2} \, \mathrm{d}\widetilde{\mu}(s_1) & \cdots & \int_{\mathbb{R}} s_1^{1} \, \mathrm{d}\widetilde{\mu}(s_1) \\ \vdots & \vdots & \ddots & \vdots \\ \int_{\mathbb{R}} s_{2n-1}^{-1} \, \mathrm{d}\widetilde{\mu}(s_{2n-1}) & \int_{\mathbb{R}} s_{2n-1}^{0} \, \mathrm{d}\widetilde{\mu}(s_{2n-1}) & \cdots & \int_{\mathbb{R}} s_{2n-1}^{2n-1} \, \mathrm{d}\widetilde{\mu}(s_{2n-1}) \\ z^0 & z^1 & \cdots & z^{2n} \end{vmatrix} \\
&= \frac{z^{-n}}{H_{2n}^{(-2n)}} \underbrace{\int_{\mathbb{R}} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}}}_{2n} \mathrm{d}\widetilde{\mu}(s_0) \mathrm{d}\widetilde{\mu}(s_1) \cdots \mathrm{d}\widetilde{\mu}(s_{2n-1}) \begin{vmatrix} s_0^{-2n} & s_0^{-2n+1} & \cdots & s_0^{-1} & s_0^0 \\ s_1^{-2n+1} & s_1^{-2n+2} & \cdots & s_1^0 & s_1^1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ s_{2n-1}^{-1} & s_{2n-1}^{0} & \cdots & s_{2n-1}^{2n-2} & s_{2n-1}^{2n-1} \\ z^0 & z^1 & \cdots & z^{2n-1} & z^{2n} \end{vmatrix} \\
&= \frac{z^{-n}}{H_{2n}^{(-2n)}} \underbrace{\int_{\mathbb{R}} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}}}_{2n} \mathrm{d}\widetilde{\mu}(s_0) \mathrm{d}\widetilde{\mu}(s_1) \cdots \mathrm{d}\widetilde{\mu}(s_{2n-1}) s_0^{-2n} s_1^{-2n+1} \cdots s_{2n-1}^{-1} \\
&\times \begin{vmatrix} s_0^0 & s_0^1 & \cdots & s_0^{2n-1} & s_0^{2n} \\ s_1^0 & s_1^1 & \cdots & s_1^{2n-1} & s_1^{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ s_{2n-1}^0 & s_{2n-1}^1 & \cdots & s_{2n-1}^{2n-1} & s_{2n-1}^{2n} \\ z^0 & z^1 & \cdots & z^{2n-1} & z^{2n} \end{vmatrix} \\
&= \frac{z^{-n}}{H_{2n}^{(-2n)} (2n)!} \sum_{\boldsymbol{\sigma} \in \mathfrak{S}_{2n}} \underbrace{\int_{\mathbb{R}} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}}}_{2n} \mathrm{d}\widetilde{\mu}(s_{\sigma(0)}) \mathrm{d}\widetilde{\mu}(s_{\sigma(1)}) \cdots \mathrm{d}\widetilde{\mu}(s_{\sigma(2n-1)}) s_{\sigma(0)}^{-2n} s_{\sigma(1)}^{-2n} \cdots s_{\sigma(2n-1)}^{-2n} \\
&\times s_{\sigma(0)}^0 s_{\sigma(1)}^1 \cdots s_{\sigma(2n-1)}^{2n-1} \begin{vmatrix} s_{\sigma(0)}^0 & s_{\sigma(0)}^1 & \cdots & s_{\sigma(0)}^{2n-1} & s_{\sigma(0)}^{2n} \\ s_{\sigma(1)}^0 & s_{\sigma(1)}^1 & \cdots & s_{\sigma(1)}^{2n-1} & s_{\sigma(1)}^{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ s_{\sigma(2n-1)}^0 & s_{\sigma(2n-1)}^1 & \cdots & s_{\sigma(2n-1)}^{2n-1} & s_{\sigma(2n-1)}^{2n} \\ z^0 & z^1 & \cdots & z^{2n-1} & z^{2n} \end{vmatrix}
\end{aligned}$$

$$
=\frac{z^{-n}}{H_{2n}^{(-2n)}(2n)!}\underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}}_{2n}\mathrm{d}\widetilde{\mu}(s_0)\mathrm{d}\widetilde{\mu}(s_1)\cdots\mathrm{d}\widetilde{\mu}(s_{2n-1})s_0^{-2n}s_1^{-2n}\cdots s_{2n-1}^{-2n}
$$

$$
\times\left(\sum_{\sigma\in\mathfrak{S}_{2n}}\operatorname{sgn}(\boldsymbol{\sigma})s_{\sigma(0)}^0s_{\sigma(1)}^1\cdots s_{\sigma(2n-1)}^{2n-1}\right)\begin{vmatrix} s_0^0 & s_0^1 & \cdots & s_0^{2n-1} & s_0^{2n}\\ s_1^0 & s_1^1 & \cdots & s_1^{2n-1} & s_1^{2n}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ s_{2n-1}^0 & s_{2n-1}^1 & \cdots & s_{2n-1}^{2n-1} & s_{2n-1}^{2n}\\ z^0 & z^1 & \cdots & z^{2n-1} & z^{2n}\end{vmatrix}
$$

$$
=\frac{z^{-n}}{H_{2n}^{(-2n)}(2n)!}\underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}}_{2n}\mathrm{d}\widetilde{\mu}(s_0)\mathrm{d}\widetilde{\mu}(s_1)\cdots\mathrm{d}\widetilde{\mu}(s_{2n-1})s_0^{-2n}s_1^{-2n}\cdots s_{2n-1}^{-2n}
$$

$$
\times\begin{vmatrix} s_0^0 & s_0^1 & \cdots & s_0^{2n-1}\\ s_1^0 & s_1^1 & \cdots & s_1^{2n-1}\\ \vdots & \vdots & \ddots & \vdots\\ s_{2n-1}^0 & s_{2n-1}^1 & \cdots & s_{2n-1}^{2n-1}\end{vmatrix}\begin{vmatrix} s_0^0 & s_0^1 & \cdots & s_0^{2n-1} & s_0^{2n}\\ s_1^0 & s_1^1 & \cdots & s_1^{2n-1} & s_1^{2n}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ s_{2n-1}^0 & s_{2n-1}^1 & \cdots & s_{2n-1}^{2n-1} & s_{2n-1}^{2n}\\ z^0 & z^1 & \cdots & z^{2n-1} & z^{2n}\end{vmatrix};
$$

but a straightforward calculation shows that

$$
\begin{vmatrix} s_0^0 & s_0^1 & \cdots & s_0^{2n-1} & s_0^{2n}\\ s_1^0 & s_1^1 & \cdots & s_1^{2n-1} & s_1^{2n}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ s_{2n-1}^0 & s_{2n-1}^1 & \cdots & s_{2n-1}^{2n-1} & s_{2n-1}^{2n}\\ z^0 & z^1 & \cdots & z^{2n-1} & z^{2n}\end{vmatrix}=\begin{vmatrix} s_0^0 & s_0^1 & \cdots & s_0^{2n-1}\\ s_1^0 & s_1^1 & \cdots & s_1^{2n-1}\\ \vdots & \vdots & \ddots & \vdots\\ s_{2n-1}^0 & s_{2n-1}^1 & \cdots & s_{2n-1}^{2n-1}\end{vmatrix}\prod_{j=0}^{2n-1}(z-s_j),
$$

whence

$$
\boldsymbol{\pi}_{2n}(z)=\frac{z^{-n}}{H_{2n}^{(-2n)}(2n)!}\underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}}_{2n}\mathrm{d}\widetilde{\mu}(s_0)\mathrm{d}\widetilde{\mu}(s_1)\cdots\mathrm{d}\widetilde{\mu}(s_{2n-1})s_0^{-2n}s_1^{-2n}\cdots s_{2n-1}^{-2n}
$$

$$
\times\prod_{j=0}^{2n-1}(z-s_j)\begin{vmatrix} s_0^0 & s_0^1 & \cdots & s_0^{2n-1}\\ s_1^0 & s_1^1 & \cdots & s_1^{2n-1}\\ \vdots & \vdots & \ddots & \vdots\\ s_{2n-1}^0 & s_{2n-1}^1 & \cdots & s_{2n-1}^{2n-1}\end{vmatrix}^2
$$

$$
=\frac{z^{-n}}{H_{2n}^{(-2n)}(2n)!}\underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}}_{2n}\mathrm{d}\widetilde{\mu}(s_0)\mathrm{d}\widetilde{\mu}(s_1)\cdots\mathrm{d}\widetilde{\mu}(s_{2n-1})s_0^{-2n}s_1^{-2n}\cdots s_{2n-1}^{-2n}
$$

$$
\times\prod_{j=0}^{2n-1}(z-s_j)\underbrace{\begin{vmatrix} 1 & 1 & \cdots & 1\\ s_0^1 & s_1^1 & \cdots & s_{2n-1}^1\\ \vdots & \vdots & \ddots & \vdots\\ s_0^{2n-1} & s_1^{2n-1} & \cdots & s_{2n-1}^{2n-1}\end{vmatrix}^2}_{=\prod_{\substack{i,l=0\\ l<i}}^{2n-1}(s_i-s_l)^2};
$$

hence the integral representation for $\boldsymbol{\pi}_{2n}(z)$ stated in the Proposition, with the integral representation for $H_{2n}^{(-2n)}$ derived in the proof of Lemma 2.2.1. See [51], Proposition 2.2.1, for the proof of the integral representation for the odd degree monic OLP $\boldsymbol{\pi}_{2n+1}(z)$. □

Remark 2.2.1. For the purposes of the ensuing asymptotic analysis, it is convenient to re-write $\mathrm{d}\widetilde{\mu}(z)=\exp(-\mathcal{N}\,V(z))\,\mathrm{d}z=\exp(-n\widetilde{V}(z))\,\mathrm{d}z=:\mathrm{d}\mu(z)$, $n\in\mathbb{N}$, where

$$
\widetilde{V}(z)=z_o V(z),
$$

with

$$z_o\colon \mathbb{N}\times\mathbb{N}\to\mathbb{R}_+,\ (\mathcal{N},n)\mapsto z_o:=\mathcal{N}/n,$$

and where the 'scaled' external field $\widetilde{V}\colon \mathbb{R}\setminus\{0\}\to\mathbb{R}$ satisfies the following conditions:

$$\widetilde{V} \text{ is real analytic on } \mathbb{R}\setminus\{0\}; \tag{2.3}$$

$$\lim_{|x|\to\infty}\left(\widetilde{V}(x)/\ln(x^2+1)\right)=+\infty; \tag{2.4}$$

$$\lim_{|x|\to 0}\left(\widetilde{V}(x)/\ln(x^{-2}+1)\right)=+\infty. \tag{2.5}$$

(For example, a rational function of the form $\widetilde{V}(z)=\sum_{k=-2m_1}^{2m_2}\widetilde{\varrho}_k z^k$, with $\widetilde{\varrho}_k\in\mathbb{R}$, $k=-2m_1,\dots,2m_2$, $m_{1,2}\in\mathbb{N}$, and $\widetilde{\varrho}_{-2m_1},\widetilde{\varrho}_{2m_2}>0$ would satisfy conditions (2.3)–(2.5).)

Hereafter, the double-scaling limit as $\mathcal{N},n\to\infty$ such that $z_o=1+o(1)$ is studied (the simplified 'notation' $n\to\infty$ will be adopted). ■

It is, by now, a well-known, if not established, mathematical fact that variational conditions for minimisation problems in logarithmic potential theory, via the *equilibrium measure* [55, 56, 90–92], play a crucial rôle in the asymptotic analysis of (matrix) RHPs associated with (continuous and discrete) orthogonal polynomials, their roots, and corresponding recurrence relation coefficients (see, for example, [58, 59, 61, 65, 75]). The situation with respect to the large-n asymptotic analysis for the monic OLPs, $\boldsymbol{\pi}_n(z)$, is analogous; but, unlike the asymptotic analysis for the orthogonal polynomials case, the asymptotic analysis for $\boldsymbol{\pi}_n(z)$ requires the consideration of two different families of RHPs, one for even degree (**RHP1**) and one for odd degree (**RHP2**). Thus, one must consider two sets of variational conditions for two (suitably posed) minimisation problems.

The following discussion is decomposed into two parts: one part corresponding to the RHP for $\overset{e}{\mathrm{Y}}\colon \mathbb{C}\setminus\mathbb{R}\to\mathrm{SL}_2(\mathbb{C})$ formulated as **RHP1**, denoted by $\mathbf{P_1}$, and the other part corresponding to the RHP for $\overset{o}{\mathrm{Y}}\colon \mathbb{C}\setminus\mathbb{R}\to\mathrm{SL}_2(\mathbb{C})$ formulated as **RHP2**, denoted by $\mathbf{P_2}$.

$\boxed{\mathbf{P_1}}$ Let $\widetilde{V}\colon \mathbb{R}\setminus\{0\}\to\mathbb{R}$ satisfy conditions (2.3)–(2.5). Let $\mathrm{I}_V^e[\mu^e]\colon \mathcal{M}_1(\mathbb{R})\to\mathbb{R}$ denote the functional

$$\mathrm{I}_V^e[\mu^e]=\iint_{\mathbb{R}^2}\ln\left(\frac{|st|}{|s-t|^2}\right)\mathrm{d}\mu^e(s)\,\mathrm{d}\mu^e(t)+2\int_{\mathbb{R}}\widetilde{V}(s)\,\mathrm{d}\mu^e(s),$$

and consider the associated minimisation problem,

$$E_V^e=\inf\{\mathrm{I}_V^e[\mu^e];\ \mu^e\in\mathcal{M}_1(\mathbb{R})\}.$$

The infimum is finite, and there exists a unique measure μ_V^e, referred to as the 'even' equilibrium measure, achieving the infimum (that is, $\mathcal{M}_1(\mathbb{R})\ni\mu_V^e=\inf\{\mathrm{I}_V^e[\mu^e];\ \mu^e\in\mathcal{M}_1(\mathbb{R})\}$). Furthermore, μ_V^e has the following 'regularity' properties (all of these results are proven in this work):

- the 'even' equilibrium measure has compact support which consists of the disjoint union of a finite number of bounded real intervals; in fact, as shown in Section 3 (see Lemma 3.5), $\mathrm{supp}(\mu_V^e)=:J_e$[3] $=\cup_{j=1}^{N+1}(b_{j-1}^e,a_j^e)$ $(\subset\mathbb{R}\setminus\{0\})$, where $\{b_{j-1}^e,a_j^e\}_{j=1}^{N+1}$, with $b_0^e:=\min\{\mathrm{supp}(\mu_V^e)\}\notin\{-\infty,0\}$, $a_{N+1}^e:=\max\{\mathrm{supp}(\mu_V^e)\}\notin\{0,+\infty\}$, and $-\infty<b_0^e<a_1^e<b_1^e<a_2^e<\cdots<b_N^e<a_{N+1}^e<+\infty$, constitute the end-points of the support of μ_V^e;
- the end-points $\{b_{j-1}^e,a_j^e\}_{j=1}^{N+1}$ are not arbitrary; rather, they satisfy the n-dependent and (locally) solvable system of $2(N+1)$ *moment conditions* (transcendental equations) given in Lemma 3.5;
- the 'even' equilibrium measure is absolutely continuous with respect to Lebesgue measure. The *density* is given by

$$\mathrm{d}\mu_V^e(x):=\psi_V^e(x)\,\mathrm{d}x=\frac{1}{2\pi\mathrm{i}}(R_e(x))_+^{1/2}h_V^e(x)\mathbf{1}_{J_e}(x)\,\mathrm{d}x,$$

[3] It would be more usual, from the outset, for the bounded (and closed) set $\overline{J_e}:=\cup_{j=1}^{N+1}[b_{j-1}^e,a_j^e]$ to denote the support of μ_V^e; however, the open (and bounded) set J_e provides an effective description of (the interior of) the support of μ_V^e: for this reason, J_e (and at other times $\overline{J_e}$) is used to denote $\mathrm{supp}(\mu_V^e)$; *mutatis mutandis* for J_o and $\overline{J_o}$ (see $\boxed{\mathrm{P}_2}$ below). This should not cause confusion for the reader.

where

$$(R_e(z))^{1/2} := \left(\prod_{j=1}^{N+1}(z-b_{j-1}^e)(z-a_j^e)\right)^{1/2},$$

with $(R_e(x))_{\pm}^{1/2} := \lim_{\varepsilon\downarrow 0}(R_e(x\pm \mathrm{i}\varepsilon))^{1/2}$ and the branch of the square root is chosen, as per the discussion in Subsection 2.1, such that $z^{-(N+1)}(R_e(z))^{1/2} \sim_{\substack{z\to\infty \\ z\in\mathbb{C}_{\pm}}} \pm 1$, $h_V^e(z) := \frac{1}{2}\oint_{C_{\mathrm{R}}^e}(R_e(s))^{-1/2}(\frac{\mathrm{i}}{\pi s} + \frac{\mathrm{i}\widetilde{V}'(s)}{2\pi})(s-z)^{-1}\,\mathrm{d}s$ (real analytic for $z\in\mathbb{R}\setminus\{0\}$), where $'$ denotes differentiation with respect to the argument, C_{R}^e $(\subset \mathbb{C}^*)$ is the union of two circular contours, one outer one of large radius $R^{\natural}$ traversed clockwise and one inner one of small radius $r^{\natural}$ traversed counter-clockwise, with the numbers $0<r^{\natural}<R^{\natural}<+\infty$ chosen such that, for (any) non-real z in the domain of analyticity of $\widetilde{V}$ (that is, $\mathbb{C}^*$), $\mathrm{int}(C_{\mathrm{R}}^e)\supset J_e\cup\{z\}$, and $\mathbf{1}_{J_e}(x)$ denotes the indicator (characteristic) function of the set J_e. (Note that $\psi_V^e(x)\geqslant 0\ \forall\ x\in\overline{J_e} := \cup_{j=1}^{N+1}[b_{j-1}^e,a_j^e]$: it vanishes like a square root at the end-points of the support of the 'even' equilibrium measure, that is, $\psi_V^e(s) =_{s\downarrow b_{j-1}^e} \mathcal{O}((s-b_{j-1}^e)^{1/2})$ and $\psi_V^e(s) =_{s\uparrow a_j^e} \mathcal{O}((a_j^e-s)^{1/2})$, $j=1,\dots,N+1$.);

- the 'even' equilibrium measure and its (compact) support are uniquely characterised by the following Euler-Lagrange variational equations: there exists $\ell_e\in\mathbb{R}$, the 'even' Lagrange multiplier, and $\mu^e\in\mathcal{M}_1(\mathbb{R})$ such that

$$4\int_{J_e}\ln(|x-s|)\,\mathrm{d}\mu^e(s) - 2\ln|x| - \widetilde{V}(x) - \ell_e = 0, \quad x\in\overline{J_e}, \tag{P$_1^{(a)}$}$$

$$4\int_{J_e}\ln(|x-s|)\,\mathrm{d}\mu^e(s) - 2\ln|x| - \widetilde{V}(x) - \ell_e \leqslant 0, \quad x\in\mathbb{R}\setminus\overline{J_e}; \tag{P$_1^{(b)}$}$$

- the Euler-Lagrange variational equations can be conveniently recast in terms of the complex potential $g^e(z)$ of μ_V^e:

$$g^e(z) := \int_{J_e}\ln\Big((z-s)^2(zs)^{-1}\Big)\,\mathrm{d}\mu_V^e(s), \quad z\in\mathbb{C}\setminus(-\infty,\max\{0,a_{N+1}^e\}).$$

The function $g^e\colon \mathbb{C}\setminus(-\infty,\max\{0,a_{N+1}^e\})\to\mathbb{C}$ so defined satisfies:
(P$_1^{(1)}$) $g^e(z)$ is analytic for $z\in\mathbb{C}\setminus(-\infty,\max\{0,a_{N+1}^e\})$;
(P$_1^{(2)}$) $g^e(z) =_{\substack{z\to\infty \\ z\in\mathbb{C}\setminus\mathbb{R}}} \ln(z) + \mathcal{O}(1)$;
(P$_1^{(3)}$) $g_+^e(z) + g_-^e(z) - \widetilde{V}(z) - \ell_e + 2Q_e = 0$, $z\in\overline{J_e}$, where $g_{\pm}^e(z) := \lim_{\varepsilon\downarrow 0} g^e(z\pm\mathrm{i}\varepsilon)$, and $Q_e := \int_{J_e}\ln(s)\,\mathrm{d}\mu_V^e(s) = \int_{J_e}\ln(|s|)\,\mathrm{d}\mu_V^e(s) + \mathrm{i}\pi\int_{J_e\cap\mathbb{R}_-}\mathrm{d}\mu_V^e(s)$;
(P$_1^{(4)}$) $g_+^e(z) + g_-^e(z) - \widetilde{V}(z) - \ell_e + 2Q_e \leqslant 0$, $z\in\mathbb{R}\setminus\overline{J_e}$, where equality holds for at most a finite number of points;
(P$_1^{(5)}$) $g_+^e(z) - g_-^e(z) = \mathrm{i}f_{g^e}^{\mathbb{R}}(z)$, $z\in\mathbb{R}$, where $f_{g^e}^{\mathbb{R}}\colon \mathbb{R}\to\mathbb{R}$, and, in particular, $g_+^e(z)-g_-^e(z) = \mathrm{i}$ const., $z\in\mathbb{R}\setminus\overline{J_e}$, with const. $\in\mathbb{R}$;
(P$_1^{(6)}$) $\mathrm{i}(g_+^e(z)-g_-^e(z))'\geqslant 0$, $z\in J_e$, where equality holds for at most a finite number of points.

P$_2$ Let $\widetilde{V}\colon\mathbb{R}\setminus\{0\}\to\mathbb{R}$ satisfy conditions (2.3)–(2.5). Let $\mathrm{I}_V^o[\mu^o]\colon \mathcal{M}_1(\mathbb{R})\to\mathbb{R}$ denote the functional

$$\mathrm{I}_V^o[\mu^o] = \iint_{\mathbb{R}^2}\ln\left(\frac{|st|}{|s-t|^{2+\frac{1}{n}}}\right)\mathrm{d}\mu^o(s)\,\mathrm{d}\mu^o(t) + 2\int_{\mathbb{R}}\widetilde{V}(s)\,\mathrm{d}\mu^o(s), \quad n\in\mathbb{N},$$

and consider the associated minimisation problem,

$$E_V^o = \inf\{\mathrm{I}_V^o[\mu^o];\ \mu^o\in\mathcal{M}_1(\mathbb{R})\}.$$

The infimum is finite, and there exists a unique measure μ_V^o, referred to as the 'odd' equilibrium measure, achieving the infimum (that is, $\mathcal{M}_1(\mathbb{R})\ni\mu_V^o = \inf\{\mathrm{I}_V^o[\mu^o];\ \mu^o\in\mathcal{M}_1(\mathbb{R})\}$). Furthermore, μ_V^o has the following 'regularity' properties (see [51] for complete details and proofs):

- the 'odd' equilibrium measure has compact support which consists of the disjoint union of a finite number of bounded real intervals; in fact, as shown in [51], $\mathrm{supp}(\mu_V^o) =: J_o = \cup_{j=1}^{N+1}(b_{j-1}^o, a_j^o)$ $(\subset \mathbb{R} \setminus \{0\})$, where $\{b_{j-1}^o, a_j^o\}_{j=1}^{N+1}$, with $b_0^o := \min\{\mathrm{supp}(\mu_V^o)\} \notin \{-\infty, 0\}$, $a_{N+1}^o := \max\{\mathrm{supp}(\mu_V^o)\} \notin \{0, +\infty\}$, and $-\infty < b_0^o < a_1^o < b_1^o < a_2^o < \cdots < b_N^o < a_{N+1}^o < +\infty$, constitute the end-points of the support of μ_V^o; (The number of intervals, $N+1$, is the same in the 'odd' case as in the 'even' case, which can be established by a lengthy analysis similar to that contained in [92].)
- the end-points $\{b_{j-1}^o, a_j^o\}_{j=1}^{N+1}$ are not arbitrary; rather, they satisfy an n-dependent and (locally) solvable system of $2(N+1)$ moment conditions (transcendental equations; see [51], Lemma 3.5);
- the 'odd' equilibrium measure is absolutely continuous with respect to Lebesgue measure. The density is given by

$$\mathrm{d}\mu_V^o(x) := \psi_V^o(x)\,\mathrm{d}x = \frac{1}{2\pi\mathrm{i}}(R_o(x))_+^{1/2} h_V^o(x) \mathbf{1}_{J_o}(x)\,\mathrm{d}x,$$

where

$$(R_o(z))^{1/2} := \left(\prod_{j=1}^{N+1}(z-b_{j-1}^o)(z-a_j^o)\right)^{1/2},$$

with $(R_o(x))_\pm^{1/2} := \lim_{\varepsilon\downarrow 0}(R_o(x \pm \mathrm{i}\varepsilon))^{1/2}$ and the branch of the square root is chosen, as per the discussion in Subsection 2.1, such that $z^{-(N+1)}(R_o(z))^{1/2} \sim_{\substack{z\to\infty \\ z\in\mathbb{C}_\pm}} \pm 1$, $h_V^o(z) := (2+\frac{1}{n})^{-1}\oint_{C_{\mathrm{R}}^o}(R_o(s))^{-1/2}(\frac{\mathrm{i}}{\pi s} + \frac{\mathrm{i}\widetilde{V}'(s)}{2\pi})(s-z)^{-1}\,\mathrm{d}s$ (real analytic for $z \in \mathbb{R} \setminus \{0\}$), where C_{R}^o $(\subset \mathbb{C}^*)$ is the union of two circular contours, one outer one of large radius $R^\flat$ traversed clockwise and one inner one of small radius $r^\flat$ traversed counter-clockwise, with the numbers $0 < r^\flat < R^\flat < +\infty$ chosen such that, for (any) non-real z in the domain of analyticity of $\widetilde{V}$ (that is, $\mathbb{C}^*$), $\mathrm{int}(C_{\mathrm{R}}^o) \supset J_o \cup \{z\}$, and $\mathbf{1}_{J_o}(x)$ denotes the indicator (characteristic) function of the set J_o. (Note that $\psi_V^o(x) \geqslant 0\ \forall\ x \in \overline{J_o} := \cup_{j=1}^{N+1}[b_{j-1}^o, a_j^o]$: it vanishes like a square root at the end-points of the support of the 'odd' equilibrium measure, that is, $\psi_V^o(s) =_{s\downarrow b_{j-1}^o} \mathcal{O}((s-b_{j-1}^o)^{1/2})$ and $\psi_V^o(s) =_{s\uparrow a_j^o} \mathcal{O}((a_j^o - s)^{1/2})$, $j = 1, \dots, N+1$.);
- the 'odd' equilibrium measure and its (compact) support are uniquely characterised by the following Euler-Lagrange variational equations: there exists $\ell_o \in \mathbb{R}$, the 'odd' Lagrange multiplier, and $\mu^o \in \mathcal{M}_1(\mathbb{R})$ such that

$$2\left(2+\frac{1}{n}\right)\int_{J_o}\ln(|x-s|)\,\mathrm{d}\mu^o(s) - 2\ln|x| - \widetilde{V}(x) - \ell_o - 2\left(2+\frac{1}{n}\right)\widetilde{Q}_o = 0, \quad x \in \overline{J_o}, \tag{$\mathrm{P}_2^{(a)}$}$$

$$2\left(2+\frac{1}{n}\right)\int_{J_o}\ln(|x-s|)\,\mathrm{d}\mu^o(s) - 2\ln|x| - \widetilde{V}(x) - \ell_o - 2\left(2+\frac{1}{n}\right)\widetilde{Q}_o \leqslant 0, \quad x \in \mathbb{R} \setminus \overline{J_o}, \tag{$\mathrm{P}_2^{(b)}$}$$

where $\widetilde{Q}_o := \int_{J_o}\ln(|s|)\,\mathrm{d}\mu^o(s)$;
- the Euler-Lagrange variational equations can be conveniently recast in terms of the complex potential $g^o(z)$ of μ_V^o:

$$g^o(z) := \int_{J_o}\ln\left((z-s)^{2+\frac{1}{n}}(zs)^{-1}\right)\mathrm{d}\mu_V^o(s), \quad z \in \mathbb{C} \setminus (-\infty, \max\{0, a_{N+1}^o\}).$$

The function $g^o\colon \mathbb{C} \setminus (-\infty, \max\{0, a_{N+1}^o\}) \to \mathbb{C}$ so defined satisfies:

($\mathrm{P}_2^{(1)}$) $g^o(z)$ is analytic for $z \in \mathbb{C} \setminus (-\infty, \max\{0, a_{N+1}^o\})$;

($\mathrm{P}_2^{(2)}$) $g^o(z) =_{\substack{z\to 0 \\ z\in\mathbb{C}\setminus\mathbb{R}}} -\ln(z) + \mathcal{O}(1)$;

($\mathrm{P}_2^{(3)}$) $g_+^o(z) + g_-^o(z) - \widetilde{V}(z) - \ell_o - \mathfrak{Q}_{\mathcal{A}}^+ - \mathfrak{Q}_{\mathcal{A}}^- = 0$, $z \in \overline{J_o}$, where $g_\pm^o(z) := \lim_{\varepsilon\downarrow 0} g^o(z \pm \mathrm{i}\varepsilon)$, and $\mathfrak{Q}_{\mathcal{A}}^\pm := (1+\frac{1}{n})\int_{J_o}\ln(|s|)\,\mathrm{d}\mu_V^o(s) - \mathrm{i}\pi\int_{J_o\cap\mathbb{R}_-}\mathrm{d}\mu_V^o(s) \pm \mathrm{i}\pi(2+\frac{1}{n})\int_{J_o\cap\mathbb{R}_+}\mathrm{d}\mu_V^o(s)$;

($\mathrm{P}_2^{(4)}$) $g_+^o(z) + g_-^o(z) - \widetilde{V}(z) - \ell_o - \mathfrak{Q}_{\mathcal{A}}^+ - \mathfrak{Q}_{\mathcal{A}}^- \leqslant 0$, $z \in \mathbb{R} \setminus \overline{J_o}$, where equality holds for at most a finite number of points;

($\mathrm{P}_2^{(5)}$) $g^o_+(z)-g^o_-(z)-\mathfrak{Q}^+_{\mathcal{A}}+\mathfrak{Q}^-_{\mathcal{A}}=\mathrm{i}f^{\mathbb{R}}_{g^o}(z)$, $z\in\mathbb{R}$, where $f^{\mathbb{R}}_{g^o}\colon\mathbb{R}\to\mathbb{R}$, and, in particular, $g^o_+(z)-g^o_-(z)-\mathfrak{Q}^+_{\mathcal{A}}+\mathfrak{Q}^-_{\mathcal{A}}=\mathrm{i}\,\text{const.}$, $z\in\mathbb{R}\setminus\overline{J_o}$, with const.$\in\mathbb{R}$;

($\mathrm{P}_2^{(6)}$) $\mathrm{i}(g^o_+(z)-g^o_-(z)-\mathfrak{Q}^+_{\mathcal{A}}+\mathfrak{Q}^-_{\mathcal{A}})'\geqslant 0$, $z\in J_o$, where equality holds for at most a finite number of points.

In this three-fold series of works on asymptotics of OLPs and related quantities, the so-called 'regular case' is studied, namely:

- $\mathrm{d}\mu^e_V$, or $\widetilde{V}\colon\mathbb{R}\setminus\{0\}\to\mathbb{R}$ satisfying conditions (2.3)–(2.5), is *regular* if: (i) $h^e_V(x)\not\equiv 0$ on $\overline{J_e}$; (ii) $4\int_{J_e}\ln(|x-s|)\,\mathrm{d}\mu^e_V(s)-2\ln|x|-\widetilde{V}(x)-\ell_e<0$, $x\in\mathbb{R}\setminus\overline{J_e}$; and (iii) inequalities ($\mathrm{P}_1^{(4)}$) and ($\mathrm{P}_1^{(6)}$) in $\mathbf{P_1}$ are strict, that is, $\leqslant$ (resp., $\geqslant$) is replaced by $<$ (resp., $>$);
- $\mathrm{d}\mu^o_V$, or $\widetilde{V}\colon\mathbb{R}\setminus\{0\}\to\mathbb{R}$ satisfying conditions (2.3)–(2.5), is regular if: (i) $h^o_V(x)\not\equiv 0$ on $\overline{J_o}$; (ii) $2(2+\frac{1}{n})\int_{J_o}\ln(|x-s|)\,\mathrm{d}\mu^o_V(s)-2\ln|x|-\widetilde{V}(x)-\ell_o-2(2+\frac{1}{n})Q_o<0$, $x\in\mathbb{R}\setminus\overline{J_o}$, where $Q_o:=\int_{J_o}\ln(|s|)\,\mathrm{d}\mu^o_V(s)$; and (iii) inequalities ($\mathrm{P}_2^{(4)}$) and ($\mathrm{P}_2^{(6)}$) in $\mathbf{P_2}$ are strict, that is, $\leqslant$ (resp., $\geqslant$) is replaced by $<$ (resp., $>$)[4].

The (density of the) 'even' and 'odd' equilibrium measures $\mathrm{d}\mu^e_V$ and $\mathrm{d}\mu^o_V$, respectively, together with the corresponding variational problems, emerge naturally in the asymptotic analyses of **RHP1** and **RHP2**.

Remark 2.2.2. The following correspondences should also be noted:

- $g^e\colon\mathbb{C}\setminus(-\infty,\max\{0,a^e_{N+1}\})\to\mathbb{C}$ solves the *phase conditions* ($\mathrm{P}_1^{(1)}$)–($\mathrm{P}_1^{(6)}$) $\Leftrightarrow\mathcal{M}_1(\mathbb{R})\ni\mu^e_V$ solves the variational conditions ($\mathrm{P}_1^{(a)}$) and ($\mathrm{P}_1^{(b)}$);
- $g^o\colon\mathbb{C}\setminus(-\infty,\max\{0,a^o_{N+1}\})\to\mathbb{C}$ solves the phase conditions ($\mathrm{P}_2^{(1)}$)–($\mathrm{P}_2^{(6)}$) $\Leftrightarrow\mathcal{M}_1(\mathbb{R})\ni\mu^o_V$ solves the variational conditions ($\mathrm{P}_2^{(a)}$) and ($\mathrm{P}_2^{(b)}$). ■

Since the main results of this paper are asymptotics (as $n\to\infty$) for $\boldsymbol{\pi}_{2n}(z)$ ($z\in\mathbb{C}$), $\xi^{(2n)}_n$ and $\phi_{2n}(z)$ ($z\in\mathbb{C}$), which are, via Lemma 2.2.1, Equation (2.2), and Equations (1.2) and (1.4), related to **RHP1** for $\overset{e}{\mathrm{Y}}\colon\mathbb{C}\setminus\mathbb{R}\to\mathrm{SL}_2(\mathbb{C})$, no further reference, henceforth, to **RHP2** (and Lemma 2.2.2) for $\overset{o}{\mathrm{Y}}\colon\mathbb{C}\setminus\mathbb{R}\to\mathrm{SL}_2(\mathbb{C})$ will be made (see [51] for the complete details of the asymptotic analysis of **RHP2**). In the ensuing analysis, the large-n behaviour of the solution of **RHP1** (see Lemma 2.2.1, Equation (2.2)), hence asymptotics for $\boldsymbol{\pi}_{2n}(z)$ (in the entire complex plane), $\xi^{(2n)}_n$ and $\phi_{2n}(z)$ (in the entire complex plane), are extracted.

2.3 Summary of Results

In this subsection, the final results of this work are presented (see Sections 3–5 for the detailed analyses and proofs). Before doing so, however, some notational preamble is necessary. For $j=1,\dots,N+1$, let

$$\Phi^e_{a_j}(z):=\left(\frac{3n}{2}\int_{a^e_j}^{z}(R_e(s))^{1/2}h^e_V(s)\,\mathrm{d}s\right)^{2/3}\quad\text{and}\quad\Phi^e_{b_{j-1}}(z):=\left(-\frac{3n}{2}\int_{z}^{b^e_{j-1}}(R_e(s))^{1/2}h^e_V(s)\,\mathrm{d}s\right)^{2/3},$$

where $(R_e(z))^{1/2}$ and $h^e_V(z)$ are defined in Theorem 2.3.1, Equations (2.8) and (2.9). Define the 'small', mutually disjoint open discs about the end-points of the support of the 'even' equilibrium measure, $\{b^e_{j-1},a^e_j\}_{j=1}^{N+1}$, as follows: for $j=1,\dots,N+1$,

$$\mathbb{U}^e_{\delta_{a_j}}:=\left\{z\in\mathbb{C};\ |z-a^e_j|<\delta^e_{a_j}\right\}\quad\text{and}\quad\mathbb{U}^e_{\delta_{b_{j-1}}}:=\left\{z\in\mathbb{C};\ |z-b^e_{j-1}|<\delta^e_{b_{j-1}}\right\},$$

[4]There are three distinct situations in which these conditions may fail: (i) for at least one $\widetilde{x}_e\in\mathbb{R}\setminus\overline{J_e}$ (resp., $\widetilde{x}_o\in\mathbb{R}\setminus\overline{J_o}$), $4\int_{J_e}\ln(|\widetilde{x}_e-s|)\,\mathrm{d}\mu^e_V(s)-2\ln|\widetilde{x}_e|-\widetilde{V}(\widetilde{x}_e)-\ell_e=0$ (resp., $2(2+\frac{1}{n})\int_{J_o}\ln(|\widetilde{x}_o-s|)\,\mathrm{d}\mu^o_V(s)-2\ln|\widetilde{x}_o|-\widetilde{V}(\widetilde{x}_o)-\ell_o-2(2+\frac{1}{n})Q_o=0$), that is, for n even (resp., n odd) equality is attained for at least one point $\widetilde{x}_e$ (resp., $\widetilde{x}_o$) in the complement of the closure of the support of the 'even' (resp., 'odd') equilibrium measure μ^e_V (resp., μ^o_V), which corresponds to the situation in which a 'band' has just closed, or is about to open, about $\widetilde{x}_e$ (resp., $\widetilde{x}_o$); (ii) for at least one $\widehat{x}_e$ (resp., $\widehat{x}_o$), $h^e_V(\widehat{x}_e)=0$ (resp., $h^o_V(\widehat{x}_o)=0$), that is, for n even (resp., n odd) the function h^e_V (resp., h^o_V) vanishes for at least one point $\widehat{x}_e$ (resp., $\widehat{x}_o$) within the support of the 'even' (resp., 'odd') equilibrium measure μ^e_V (resp., μ^o_V), which corresponds to the situation in which a 'gap' is about to open, or close, about $\widehat{x}_e$ (resp., $\widehat{x}_o$); and (iii) there exists at least one $j\in\{1,\dots,N+1\}$, denoted j_e (resp., j_o), such that $h^e_V(b^e_{j_e-1})=0$ and/or $h^e_V(a^e_{j_e})=0$ (resp., $h^o_V(b^o_{j_o-1})=0$ and/or $h^o_V(a^o_{j_o})=0$). Each of these three cases can occur only a finite number of times due to the fact that $\widetilde{V}\colon\mathbb{R}\setminus\{0\}\to\mathbb{R}$ satisfies conditions (2.3)–(2.5) [58, 92].

where $(0,1) \ni \delta^e_{a_j}$ (resp., $(0,1) \ni \delta^e_{b_{j-1}}$) are chosen 'sufficiently small' so that $\Phi^e_{a_j}(z)$ (resp., $\Phi^e_{b_{j-1}}(z)$), which are bi-holomorphic, conformal, and orientation preserving (resp., bi-holomorphic, conformal, and non-orientation preserving), map $\mathbb{U}^e_{\delta_{a_j}}$ (resp., $\mathbb{U}^e_{\delta_{b_{j-1}}}$), as well as the oriented skeletons (see Figure 5) $\cup_{l=1}^4 \Sigma^{e,l}_{a_j}$ (resp., $\cup_{l=1}^4 \Sigma^{e,l}_{b_{j-1}}$ (see Figure 6)), injectively onto open (and convex), n-dependent neighbourhoods of 0 such that:

(i) $\Phi^e_{a_j}(a^e_j) = 0$ (resp., $\Phi^e_{b_{j-1}}(b^e_{j-1}) = 0$);

(ii) $\Phi^e_{a_j} \colon \mathbb{U}^e_{\delta_{a_j}} \to \widehat{\mathbb{U}}^e_{\delta_{a_j}} := \Phi^e_{a_j}(\mathbb{U}^e_{\delta_{a_j}})$ (resp., $\Phi^e_{b_{j-1}} \colon \mathbb{U}^e_{\delta_{b_{j-1}}} \to \widehat{\mathbb{U}}^e_{\delta_{b_{j-1}}} := \Phi^e_{b_{j-1}}(\mathbb{U}^e_{\delta_{b_{j-1}}})$);

(iii) $\Phi^e_{a_j}(\mathbb{U}^e_{\delta_{a_j}} \cap \Sigma^{e,l}_{a_j}) = \Phi^e_{a_j}(\mathbb{U}^e_{\delta_{a_j}}) \cap \gamma^{e,l}_{a_j}$ (resp., $\Phi^e_{b_{j-1}}(\mathbb{U}^e_{\delta_{b_{j-1}}} \cap \Sigma^{e,l}_{b_{j-1}}) = \Phi^e_{b_{j-1}}(\mathbb{U}^e_{\delta_{b_{j-1}}}) \cap \gamma^{e,l}_{b_{j-1}}$);

(iv) $\Phi^e_{a_j}(\mathbb{U}^e_{\delta_{a_j}} \cap \Omega^{e,l}_{a_j}) = \Phi^e_{a_j}(\mathbb{U}^e_{\delta_{a_j}}) \cap \widehat{\Omega}^{e,l}_{a_j}$ (resp., $\Phi^e_{b_{j-1}}(\mathbb{U}^e_{\delta_{b_{j-1}}} \cap \Omega^{e,l}_{b_{j-1}}) = \Phi^e_{b_{j-1}}(\mathbb{U}^e_{\delta_{b_{j-1}}}) \cap \widehat{\Omega}^{e,l}_{b_{j-1}}$), with $\widehat{\Omega}^{e,1}_{a_j}$ (and $\widehat{\Omega}^{e,1}_{b_{j-1}}$) $= \{\zeta \in \mathbb{C};\ \arg(\zeta) \in (0, 2\pi/3)\}$, $\widehat{\Omega}^{e,2}_{a_j}$ (and $\widehat{\Omega}^{e,2}_{b_{j-1}}$) $= \{\zeta \in \mathbb{C};\ \arg(\zeta) \in (2\pi/3, \pi)\}$, $\widehat{\Omega}^{e,3}_{a_j}$ (and $\widehat{\Omega}^{e,3}_{b_{j-1}}$) $= \{\zeta \in \mathbb{C};\ \arg(\zeta) \in (-\pi, -2\pi/3)\}$, and $\widehat{\Omega}^{e,4}_{a_j}$ (and $\widehat{\Omega}^{e,4}_{b_{j-1}}$) $= \{\zeta \in \mathbb{C};\ \arg(\zeta) \in (-2\pi/3, 0)\}$[5].

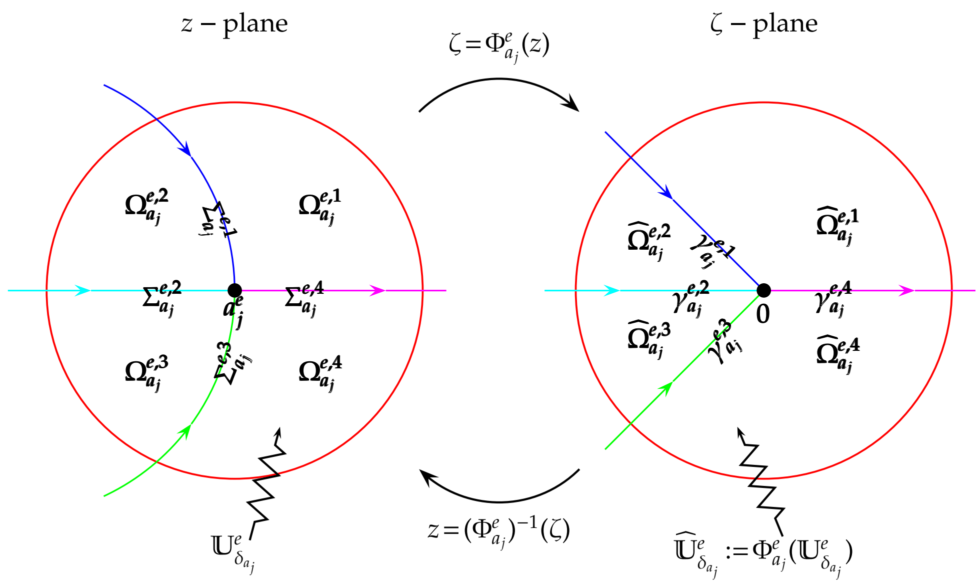

Figure 5: The conformal mapping $\zeta = \Phi^e_{a_j}(z) := (\frac{3n}{2} \int_{a^e_j}^z (R_e(s))^{1/2} h^e_V(s))^{2/3}$, $j = 1, \dots, N+1$, where $(\Phi^e_{a_j})^{-1}$ denotes the inverse mapping

Introduce, now, the Airy function, $\mathrm{Ai}(\cdot)$, which appears in several of the final results of this work: $\mathrm{Ai}(\cdot)$ is determined (uniquely) as the solution of the second-order, non-constant coefficient, homogeneous ODE (see, for example, Chapter 10 of [93])

$$\mathrm{Ai}''(z) - z\,\mathrm{Ai}(z) = 0,$$

with asymptotics (at infinity)

$$\begin{aligned} \mathrm{Ai}(z) &\underset{\substack{z \to \infty \\ |\arg z| < \pi}}{\sim} \frac{1}{2\sqrt{\pi}} z^{-1/4} \, \mathrm{e}^{-\widehat{\zeta}(z)} \sum_{k=0}^{\infty} (-1)^k s_k (\widehat{\zeta}(z))^{-k}, \qquad \widehat{\zeta}(z) := \frac{2}{3} z^{3/2}, \\ \mathrm{Ai}'(z) &\underset{\substack{z \to \infty \\ |\arg z| < \pi}}{\sim} -\frac{1}{2\sqrt{\pi}} z^{1/4} \, \mathrm{e}^{-\widehat{\zeta}(z)} \sum_{k=0}^{\infty} (-1)^k t_k (\widehat{\zeta}(z))^{-k}, \end{aligned} \tag{2.6}$$

[5] The precise angles between the sectors are not absolutely important; one could, for example, replace $2\pi/3$ by any angle strictly between 0 and π [2, 58, 59, 61, 90].

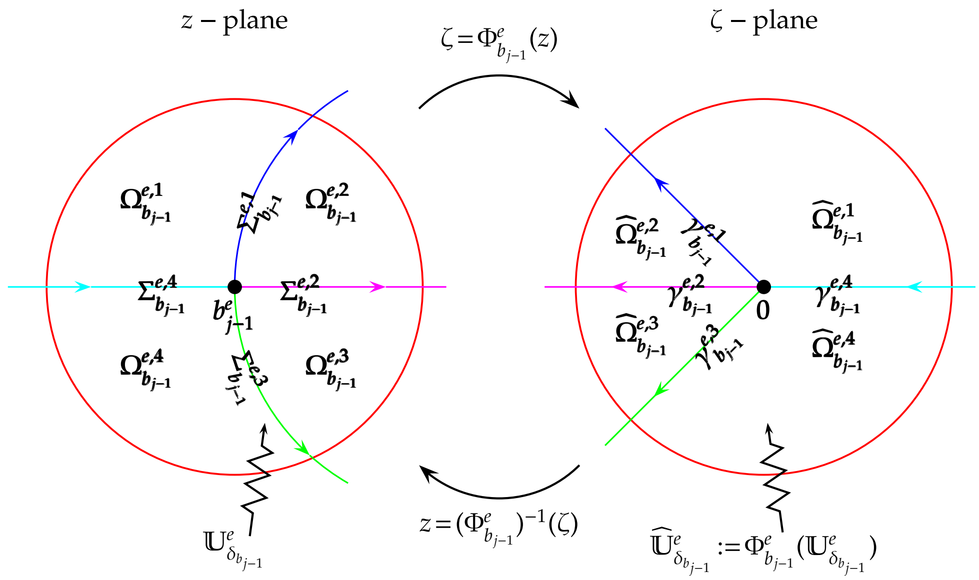

Figure 6: The conformal mapping $\zeta = \Phi^{e}_{b_{j-1}}(z) := (-\frac{3n}{2}\int_{z}^{b^{e}_{j-1}}(R_e(s))^{1/2}h^{e}_{V}(s))^{2/3}$, $j = 1,\dots,N+1$, where $(\Phi^{e}_{b_{j-1}})^{-1}$ denotes the inverse mapping

where $s_0 = t_0 = 1$,

$$s_k = \frac{\Gamma(3k+1/2)}{54^k k!\,\Gamma(k+1/2)} = \frac{(2k+1)(2k+3)\cdots(6k-1)}{216^k k!}, \qquad t_k = -\left(\frac{6k+1}{6k-1}\right)s_k, \quad k \in \mathbb{N},$$

and $\Gamma(\cdot)$ is the gamma (factorial) function.

In order to present the final asymptotic (as $n \to \infty$) results, and for arbitrary $j = 1,\dots,N{+}1$, consider the following decomposition (see Figure 7), into bounded and unbounded regions, of $\mathbb{C}$ and the neighbourhoods of the end-points b^{e}_{i-1}, a^{e}_{i}, $i = 1,\dots,N{+}1$ (as per the discussion above, $\mathbb{U}^{e}_{\delta_{b_{k-1}}} \cap \mathbb{U}^{e}_{\delta_{a_k}} = \varnothing$, $k = 1,\dots,N+1$). Asymptotics (as $n \to \infty$) for $\boldsymbol{\pi}_{2n}(z)$, with $z \in \cup_{j=1}^{4}(\Upsilon^{e}_{j} \cup (\cup_{k=1}^{N+1}(\Omega^{e,j}_{b_{k-1}} \cup \Omega^{e,j}_{a_k})))$, are now

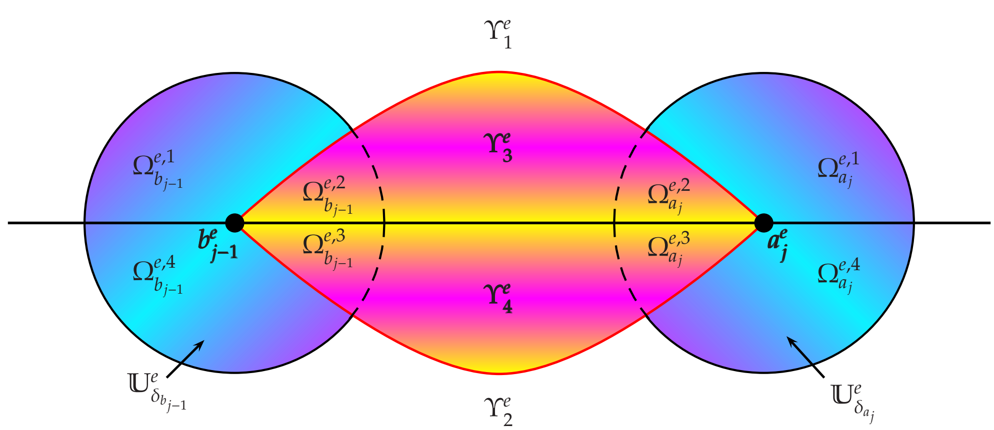

Figure 7: Region-by-region decomposition of $\mathbb{C}$ and the neighbourhoods surrounding the end-points of the support of the 'even' equilibrium measure, $\{b^{e}_{j-1}, a^{e}_{j}\}_{j=1}^{N+1}$

presented. These asymptotic expansions are obtained via a union of the DZ non-linear steepest-descent method [1,2] and the extension of Deift-Venakides-Zhou [3] (see, also, [57–76,79,94], and the

detailed pedagogical exposition [90]).

Remark 2.3.1. In order to eschew a flood of superfluous notation, the simplified 'notation' $\mathcal{O}(n^{-2})$ is maintained throughout Theorem 2.3.1 (see below), and is to be understood in the following, *normal* sense: for a compact subset, $\mathfrak{D}$, say, of $\mathbb{C}$, and uniformly with respect to $z \in \mathfrak{D}$, $\mathcal{O}(n^{-2}) := \mathcal{O}(c^{\natural}(z,n)n^{-2})$, where $\|c^{\natural}(\cdot,n)\|_{\mathcal{L}^p(\mathfrak{D})} =_{n\to\infty} \mathcal{O}(1)$, $p \in \{1,2,\infty\}$, and $\exists\ K_{\mathfrak{D}} > 0$ (and finite) such that, $\forall\ z \in \mathfrak{D}$, $|c^{\natural}(z,n)| \leqslant_{n\to\infty} K_{\mathfrak{D}}$. ∎

Theorem 2.3.1. *Let the external field* $\widetilde{V} \colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ *satisfy conditions* (2.3)–(2.5). *Set*

$$\mathrm{d}\mu_V^e(x) := \psi_V^e(x)\,\mathrm{d}x = \frac{1}{2\pi \mathrm{i}}(R_e(x))_+^{1/2} h_V^e(x) \mathbf{1}_{J_e}(x)\,\mathrm{d}x, \tag{2.7}$$

where

$$(R_e(z))^{1/2} := \left(\prod_{k=1}^{N+1}(z-b_{k-1}^e)(z-a_k^e)\right)^{1/2}, \tag{2.8}$$

with $(R_e(x))_\pm^{1/2} := \lim_{\varepsilon\downarrow 0}(R_e(x\pm \mathrm{i}\varepsilon))^{1/2}$, $x \in J_e := \operatorname{supp}(\mu_V^e) = \cup_{j=1}^{N+1}(b_{j-1}^e, a_j^e)$ $(\subset \mathbb{R}\setminus\{0\})$, $N \in \mathbb{N}$ *(and finite), where* $b_0^e := \min\{\operatorname{supp}(\mu_V^e)\} \notin \{-\infty, 0\}$, $a_{N+1}^e := \max\{\operatorname{supp}(\mu_V^e)\} \notin \{0,+\infty\}$, *and* $-\infty < b_0^e < a_1^e < b_1^e < a_2^e < \cdots < b_N^e < a_{N+1}^e < +\infty$, *the branch of the square root is chosen such that* $z^{-(N+1)}(R_e(z))^{1/2} \sim_{\substack{z\to\infty \\ z\in\mathbb{C}_\pm}} \pm 1$,

$$h_V^e(z) := \frac{1}{2}\oint_{C_{\mathrm{R}}^e} \frac{\left(\frac{\mathrm{i}}{\pi s} + \frac{\mathrm{i}\widetilde{V}'(s)}{2\pi}\right)}{\sqrt{R_e(s)}\,(s-z)}\,\mathrm{d}s \tag{2.9}$$

(real analytic for $z \in \mathbb{R}\setminus\{0\}$*),* C_{R}^e $(\subset \mathbb{C}^*)$ *is the boundary of any open doubly-connected annular region of the type* $\{z' \in \mathbb{C};\ 0 < r^{\natural} < |z'| < R^{\natural} < +\infty\}$, *where the simple outer (resp., inner) boundary* $\{z' = R^{\natural}\mathrm{e}^{\mathrm{i}\vartheta},\ 0 \leqslant \vartheta \leqslant 2\pi\}$ *(resp.,* $\{z' = r^{\natural}\mathrm{e}^{\mathrm{i}\vartheta},\ 0 \leqslant \vartheta \leqslant 2\pi\}$*) is traversed clockwise (resp., counter-clockwise), with the numbers* $0 < r^{\natural} < R^{\natural} < +\infty$ *chosen so that, for (any) non-real* z *in the domain of analyticity of* $\widetilde{V}$ *(that is,* $\mathbb{C}^*$*),* $\operatorname{int}(C_{\mathrm{R}}^e) \supset J_e \cup \{z\}$, $\mathbf{1}_{J_e}(x)$ *denotes the indicator (characteristic) function of the set* J_e, *and* $\{b_{j-1}^e, a_j^e\}_{j=1}^{N+1}$ *satisfy the following n-dependent and (locally) solvable system of* $2(N+1)$ *moment conditions:*

$$\begin{gathered}
\int_{J_e} \frac{(2s^{-1}+\widetilde{V}'(s))s^j}{(R_e(s))_+^{1/2}}\,\mathrm{d}s = 0, \quad j=0,\ldots,N, \qquad \int_{J_e} \frac{(2s^{-1}+\widetilde{V}'(s))s^{N+1}}{(R_e(s))_+^{1/2}}\,\mathrm{d}s = -4\pi\mathrm{i}, \\
\int_{a_j^e}^{b_j^e}\left(\frac{\mathrm{i}(R_e(s))^{1/2}}{2\pi}\int_{J_e}\frac{(2\xi^{-1}+\widetilde{V}'(\xi))}{(R_e(\xi))_+^{1/2}(\xi-s)}\,\mathrm{d}\xi\right)\mathrm{d}s = \ln\left|\frac{a_j^e}{b_j^e}\right| + \frac{1}{2}\left(\widetilde{V}(a_j^e)-\widetilde{V}(b_j^e)\right), \quad j=1,\ldots,N.
\end{gathered} \tag{2.10}$$

Suppose, furthermore, that $\widetilde{V} \colon \mathbb{R}\setminus\{0\} \to \mathbb{R}$ *is regular, namely:*

(i) $h_V^e(x) \not\equiv 0$ *on* $\overline{J_e} := J_e \cup \left(\cup_{k=1}^{N+1}\{b_{k-1}^e, a_k^e\}\right)$;

(ii)

$$4\int_{J_e}\ln(|x-s|)\,\mathrm{d}\mu_V^e(s) - 2\ln|x| - \widetilde{V}(x) - \ell_e = 0, \quad x \in \overline{J_e}, \tag{2.11}$$

which defines the 'even' variational constant $\ell_e \in \mathbb{R}$ *(the same on each—compact—interval* $[b_{j-1}^e, a_j^e]$, $j=1,\ldots,N+1$*), and*

$$4\int_{J_e}\ln(|x-s|)\,\mathrm{d}\mu_V^e(s) - 2\ln|x| - \widetilde{V}(x) - \ell_e < 0, \quad x \in \mathbb{R}\setminus\overline{J_e};$$

(iii)

$$g_+^e(z) + g_-^e(z) - \widetilde{V}(z) - \ell_e + 2Q_e < 0, \quad z \in \mathbb{R}\setminus\overline{J_e},$$

where

$$g^e(z) := \int_{J_e}\ln\left((z-s)^2(zs)^{-1}\right)\mathrm{d}\mu_V^e(s), \quad z \in \mathbb{C}\setminus(-\infty, \max\{0, a_{N+1}^e\}), \tag{2.12}$$

with

$$Q_e := \int_{J_e}\ln(s)\,\mathrm{d}\mu_V^e(s) = \int_{J_e}\ln(|s|)\,\mathrm{d}\mu_V^e(s) + \mathrm{i}\pi\int_{J_e\cap\mathbb{R}_-}\mathrm{d}\mu_V^e(s)$$

$$= \int_{J_e} \ln(|s|)\,\mathrm{d}\mu_V^e(s) + \mathrm{i}\pi \begin{cases} 0, & J_e \subset \mathbb{R}_+, \\ 1, & J_e \subset \mathbb{R}_-, \\ \int_{b_0^e}^{a_j^e} \mathrm{d}\mu_V^e(s), & (a_j^e, b_j^e) \ni 0, \quad j=1,\dots,N; \end{cases} \tag{2.13}$$

(iv)

$$\mathrm{i}(g_+^e(z) - g_-^e(z))' > 0, \quad z \in J_e.$$

Set

$$\overset{e}{m}{}^{\infty}(z) = \begin{cases} \overset{e}{\mathfrak{M}}{}^{\infty}(z), & z \in \mathbb{C}_+, \\ -\mathrm{i}\,\overset{e}{\mathfrak{M}}{}^{\infty}(z)\sigma_2, & z \in \mathbb{C}_-, \end{cases} \tag{2.14}$$

where $(\det(\overset{e}{m}{}^{\infty}(z)) = 1)$

$$\overset{e}{\mathfrak{M}}{}^{\infty}(z) = \begin{pmatrix} \frac{(\gamma^e(z)+(\gamma^e(z))^{-1})}{2}\mathfrak{m}_{11}^e(z) & -\frac{(\gamma^e(z)-(\gamma^e(z))^{-1})}{2\mathrm{i}}\mathfrak{m}_{12}^e(z) \\ \frac{(\gamma^e(z)-(\gamma^e(z))^{-1})}{2\mathrm{i}}\mathfrak{m}_{21}^e(z) & \frac{(\gamma^e(z)+(\gamma^e(z))^{-1})}{2}\mathfrak{m}_{22}^e(z) \end{pmatrix}, \tag{2.15}$$

$$\gamma^e(z) := \left(\left(\frac{z-b_0^e}{z-a_{N+1}^e} \right) \prod_{k=1}^{N} \left(\frac{z-b_k^e}{z-a_k^e} \right) \right)^{1/4}, \tag{2.16}$$

$$\mathfrak{m}_{11}^e(z) := \frac{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\infty)+\boldsymbol{d}_e)\boldsymbol{\theta}^e(\boldsymbol{u}^e(z)-\frac{n}{2\pi}\boldsymbol{\Omega}^e+\boldsymbol{d}_e)}{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\infty)-\frac{n}{2\pi}\boldsymbol{\Omega}^e+\boldsymbol{d}_e)\boldsymbol{\theta}^e(\boldsymbol{u}^e(z)+\boldsymbol{d}_e)}, \tag{2.17}$$

$$\mathfrak{m}_{12}^e(z) := \frac{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\infty)+\boldsymbol{d}_e)\boldsymbol{\theta}^e(-\boldsymbol{u}^e(z)-\frac{n}{2\pi}\boldsymbol{\Omega}^e+\boldsymbol{d}_e)}{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\infty)-\frac{n}{2\pi}\boldsymbol{\Omega}^e+\boldsymbol{d}_e)\boldsymbol{\theta}^e(-\boldsymbol{u}^e(z)+\boldsymbol{d}_e)}, \tag{2.18}$$

$$\mathfrak{m}_{21}^e(z) := \frac{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\infty)+\boldsymbol{d}_e)\boldsymbol{\theta}^e(\boldsymbol{u}^e(z)-\frac{n}{2\pi}\boldsymbol{\Omega}^e-\boldsymbol{d}_e)}{\boldsymbol{\theta}^e(-\boldsymbol{u}_+^e(\infty)-\frac{n}{2\pi}\boldsymbol{\Omega}^e-\boldsymbol{d}_e)\boldsymbol{\theta}^e(\boldsymbol{u}^e(z)-\boldsymbol{d}_e)}, \tag{2.19}$$

$$\mathfrak{m}_{22}^e(z) := \frac{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\infty)+\boldsymbol{d}_e)\boldsymbol{\theta}^e(-\boldsymbol{u}^e(z)-\frac{n}{2\pi}\boldsymbol{\Omega}^e-\boldsymbol{d}_e)}{\boldsymbol{\theta}^e(-\boldsymbol{u}_+^e(\infty)-\frac{n}{2\pi}\boldsymbol{\Omega}^e-\boldsymbol{d}_e)\boldsymbol{\theta}^e(\boldsymbol{u}^e(z)+\boldsymbol{d}_e)}, \tag{2.20}$$

with

$$\boldsymbol{u}^e(z) = \int_{a_{N+1}^e}^{z} \boldsymbol{\omega}^e, \qquad \boldsymbol{u}_+^e(\infty) = \int_{a_{N+1}^e}^{\infty^+} \boldsymbol{\omega}^e,$$

$\boldsymbol{\Omega}^e = (\Omega_1^e, \Omega_2^e, \dots, \Omega_N^e)^{\mathrm{T}}$ $(\in \mathbb{R}^N)$, *where*

$$\Omega_j^e := 4\pi \int_{b_j^e}^{a_{N+1}^e} \psi_V^e(s)\,\mathrm{d}s, \quad j=1,\dots,N,$$

and

$$\boldsymbol{d}_e \equiv \sum_{j=1}^{N} \int_{a_j^e}^{z_j^{e,+}} \boldsymbol{\omega}^e \quad \left(\equiv -\sum_{j=1}^{N+1} \int_{a_j^e}^{z_j^{e,-}} \boldsymbol{\omega}^e \right),$$

where

$$\left\{ z_j^{e,\pm} \right\}_{j=1}^{N} = \left\{ z^{\pm} \in \mathbb{C}_\pm;\ (\gamma^e(z) \mp (\gamma^e(z))^{-1})|_{z=z^{\pm}} = 0 \right\},$$

with $z_j^{e,\pm} \in (a_j^e, b_j^e)^{\pm}$ $(\subset \mathbb{C}_\pm)$, $j=1,\dots,N$.

Let $\overset{e}{\mathrm{Y}}\colon \mathbb{C} \setminus \mathbb{R} \to \mathrm{SL}_2(\mathbb{C})$ *be the unique solution of* **RHP1** *whose integral representations are given in Lemma* 2.2.1; *in particular,* $\boldsymbol{\pi}_{2n}(z) := (\overset{e}{\mathrm{Y}}(z))_{11}$. *Then:*
(1) *for* $z \in \Upsilon_1^e$ $(\subset \mathbb{C}_+)$,

$$\boldsymbol{\pi}_{2n}(z) \underset{n\to\infty}{=} \exp(n(g^e(z)+Q_e))\Big((\overset{e}{m}{}^{\infty}(z))_{11}\Big(1+\frac{1}{n}(\mathcal{R}_\infty^e(z))_{11}+\mathcal{O}\Big(\frac{1}{n^2}\Big)\Big) + (\overset{e}{m}{}^{\infty}(z))_{21}\Big(\frac{1}{n}(\mathcal{R}_\infty^e(z))_{12}+\mathcal{O}\Big(\frac{1}{n^2}\Big)\Big)\Big), \tag{2.21}$$

and

$$\int_{\mathbb{R}} \frac{\boldsymbol{\pi}_{2n}(s)\mathrm{e}^{-n\widetilde{V}(s)}}{s-z}\,\frac{\mathrm{d}s}{2\pi\mathrm{i}} \underset{n\to\infty}{=} \exp(-n(g^{e}(z)-\ell_{e}+Q_{e}))\Big((\overset{e}{m}{}^{\infty}(z))_{12}\Big(1+\frac{1}{n}(\mathcal{R}^{e}_{\infty}(z))_{11} + O\Big(\frac{1}{n^{2}}\Big)\Big)+(\overset{e}{m}{}^{\infty}(z))_{22}\Big(\frac{1}{n}(\mathcal{R}^{e}_{\infty}(z))_{12}+O\Big(\frac{1}{n^{2}}\Big)\Big)\Big), \tag{2.22}$$

where

$$\begin{aligned}\mathcal{R}^{e}_{\infty}(z) := {} & \sum_{j=1}^{N+1}\frac{1}{(z-b^{e}_{j-1})}\left(\frac{\mathcal{A}^{e}(b^{e}_{j-1})}{\widehat{\alpha}^{e}_{0}(b^{e}_{j-1})(z-b^{e}_{j-1})}+\frac{(\mathcal{B}^{e}(b^{e}_{j-1})\widehat{\alpha}^{e}_{0}(b^{e}_{j-1})-\mathcal{A}^{e}(b^{e}_{j-1})\widehat{\alpha}^{e}_{1}(b^{e}_{j-1}))}{(\widehat{\alpha}^{e}_{0}(b^{e}_{j-1}))^{2}}\right)\\ & +\sum_{j=1}^{N+1}\frac{1}{(z-a^{e}_{j})}\left(\frac{\mathcal{A}^{e}(a^{e}_{j})}{\widehat{\alpha}^{e}_{0}(a^{e}_{j})(z-a^{e}_{j})}+\frac{(\mathcal{B}^{e}(a^{e}_{j})\widehat{\alpha}^{e}_{0}(a^{e}_{j})-\mathcal{A}^{e}(a^{e}_{j})\widehat{\alpha}^{e}_{1}(a^{e}_{j}))}{(\widehat{\alpha}^{e}_{0}(a^{e}_{j}))^{2}}\right),\end{aligned} \tag{2.23}$$

with, for $j=1,\dots,N+1$,

$$\mathcal{A}^{e}(b^{e}_{j-1}) = -s_{1}(Q^{e}_{0}(b^{e}_{j-1}))^{-1}\mathrm{e}^{\mathrm{i}n\mho^{e}_{j-1}}\begin{pmatrix}\varkappa^{e}_{1}(b^{e}_{j-1})\varkappa^{e}_{2}(b^{e}_{j-1}) & \mathrm{i}(\varkappa^{e}_{1}(b^{e}_{j-1}))^{2}\\ \mathrm{i}(\varkappa^{e}_{2}(b^{e}_{j-1}))^{2} & -\varkappa^{e}_{1}(b^{e}_{j-1})\varkappa^{e}_{2}(b^{e}_{j-1})\end{pmatrix}, \tag{2.24}$$

$$\mathcal{A}^{e}(a^{e}_{j}) = s_{1}Q^{e}_{0}(a^{e}_{j})\mathrm{e}^{\mathrm{i}n\mho^{e}_{j}}\begin{pmatrix}-\varkappa^{e}_{1}(a^{e}_{j})\varkappa^{e}_{2}(a^{e}_{j}) & \mathrm{i}(\varkappa^{e}_{1}(a^{e}_{j}))^{2}\\ \mathrm{i}(\varkappa^{e}_{2}(a^{e}_{j}))^{2} & \varkappa^{e}_{1}(a^{e}_{j})\varkappa^{e}_{2}(a^{e}_{j})\end{pmatrix}, \tag{2.25}$$

$$\frac{\mathcal{B}^{e}(b^{e}_{j-1})}{\mathrm{e}^{\mathrm{i}n\mho^{e}_{j-1}}} = \begin{pmatrix} \boxed{\begin{gathered}\varkappa^{e}_{1}(b^{e}_{j-1})\varkappa^{e}_{2}(b^{e}_{j-1})\Big(-s_{1}(Q^{e}_{0}(b^{e}_{j-1}))^{-1}\\ \times\Big\{\daleth^{1}_{1}(b^{e}_{j-1})+\daleth^{1}_{-1}(b^{e}_{j-1})-Q^{e}_{1}(b^{e}_{j-1})\\ \times(Q^{e}_{0}(b^{e}_{j-1}))^{-1}\Big\}-t_{1}\Big\{Q^{e}_{0}(b^{e}_{j-1})\\ +(Q^{e}_{0}(b^{e}_{j-1}))^{-1}\boldsymbol{\aleph}^{1}_{1}(b^{e}_{j-1})\boldsymbol{\aleph}^{1}_{-1}(b^{e}_{j-1})\Big\}\\ +\mathrm{i}(s_{1}+t_{1})\Big\{\boldsymbol{\aleph}^{1}_{-1}(b^{e}_{j-1})-\boldsymbol{\aleph}^{1}_{1}(b^{e}_{j-1})\Big\}\Big)\end{gathered}} & \boxed{\begin{gathered}(\varkappa^{e}_{1}(b^{e}_{j-1}))^{2}\Big(-\mathrm{i}s_{1}(Q^{e}_{0}(b^{e}_{j-1}))^{-1}\Big\{2\daleth^{1}_{1}(b^{e}_{j-1})\\ -Q^{e}_{1}(b^{e}_{j-1})(Q^{e}_{0}(b^{e}_{j-1}))^{-1}\Big\}+\mathrm{i}t_{1}\Big\{Q^{e}_{0}(b^{e}_{j-1})\\ -(Q^{e}_{0}(b^{e}_{j-1}))^{-1}(\boldsymbol{\aleph}^{1}_{1}(b^{e}_{j-1}))^{2}\Big\}\\ +2(s_{1}-t_{1})\boldsymbol{\aleph}^{1}_{1}(b^{e}_{j-1})\Big)\end{gathered}} \\ \boxed{\begin{gathered}(\varkappa^{e}_{2}(b^{e}_{j-1}))^{2}\Big(-\mathrm{i}s_{1}(Q^{e}_{0}(b^{e}_{j-1}))^{-1}\Big\{2\daleth^{1}_{-1}(b^{e}_{j-1})\\ -Q^{e}_{1}(b^{e}_{j-1})(Q^{e}_{0}(b^{e}_{j-1}))^{-1}\Big\}+\mathrm{i}t_{1}\Big\{Q^{e}_{0}(b^{e}_{j-1})\\ -(Q^{e}_{0}(b^{e}_{j-1}))^{-1}(\boldsymbol{\aleph}^{1}_{-1}(b^{e}_{j-1}))^{2}\Big\}\\ -2(s_{1}-t_{1})\boldsymbol{\aleph}^{1}_{-1}(b^{e}_{j-1})\Big)\end{gathered}} & \boxed{\begin{gathered}\varkappa^{e}_{1}(b^{e}_{j-1})\varkappa^{e}_{2}(b^{e}_{j-1})\Big(s_{1}(Q^{e}_{0}(b^{e}_{j-1}))^{-1}\\ \times\Big\{\daleth^{1}_{1}(b^{e}_{j-1})+\daleth^{1}_{-1}(b^{e}_{j-1})-Q^{e}_{1}(b^{e}_{j-1})\\ \times(Q^{e}_{0}(b^{e}_{j-1}))^{-1}\Big\}+t_{1}\Big\{Q^{e}_{0}(b^{e}_{j-1})\\ +(Q^{e}_{0}(b^{e}_{j-1}))^{-1}\boldsymbol{\aleph}^{1}_{1}(b^{e}_{j-1})\boldsymbol{\aleph}^{1}_{-1}(b^{e}_{j-1})\Big\}\\ +\mathrm{i}(s_{1}+t_{1})\Big\{\boldsymbol{\aleph}^{1}_{1}(b^{e}_{j-1})-\boldsymbol{\aleph}^{1}_{-1}(b^{e}_{j-1})\Big\}\Big)\end{gathered}} \end{pmatrix}, \tag{2.26}$$

$$\frac{\mathcal{B}^{e}(a^{e}_{j})}{\mathrm{e}^{\mathrm{i}n\mho^{e}_{j}}} = \begin{pmatrix} \boxed{\begin{gathered}\varkappa^{e}_{1}(a^{e}_{j})\varkappa^{e}_{2}(a^{e}_{j})\Big(-s_{1}\Big\{Q^{e}_{1}(a^{e}_{j})\\ +Q^{e}_{0}(a^{e}_{j})\Big[\daleth^{1}_{1}(a^{e}_{j})+\daleth^{1}_{-1}(a^{e}_{j})\Big]\Big\}-t_{1}\\ \times\Big\{(Q^{e}_{0}(a^{e}_{j}))^{-1}+Q^{e}_{0}(a^{e}_{j})\boldsymbol{\aleph}^{1}_{1}(a^{e}_{j})\boldsymbol{\aleph}^{1}_{-1}(a^{e}_{j})\Big\}\\ +\mathrm{i}(s_{1}+t_{1})\Big\{\boldsymbol{\aleph}^{1}_{-1}(a^{e}_{j})-\boldsymbol{\aleph}^{1}_{1}(a^{e}_{j})\Big\}\Big)\end{gathered}} & \boxed{\begin{gathered}(\varkappa^{e}_{1}(a^{e}_{j}))^{2}\Big(\mathrm{i}s_{1}\Big\{Q^{e}_{1}(a^{e}_{j})+2Q^{e}_{0}(a^{e}_{j})\\ \times\daleth^{1}_{1}(a^{e}_{j})\Big\}+\mathrm{i}t_{1}\Big\{Q^{e}_{0}(a^{e}_{j})(\boldsymbol{\aleph}^{1}_{1}(a^{e}_{j}))^{2}\\ -(Q^{e}_{0}(a^{e}_{j}))^{-1}\Big\}-2(s_{1}-t_{1})\boldsymbol{\aleph}^{1}_{1}(a^{e}_{j})\Big)\end{gathered}} \\ \boxed{\begin{gathered}(\varkappa^{e}_{2}(a^{e}_{j}))^{2}\Big(\mathrm{i}s_{1}\Big\{Q^{e}_{1}(a^{e}_{j})+2Q^{e}_{0}(a^{e}_{j})\\ \times\daleth^{1}_{-1}(a^{e}_{j})\Big\}+\mathrm{i}t_{1}\Big\{Q^{e}_{0}(a^{e}_{j})(\boldsymbol{\aleph}^{1}_{-1}(a^{e}_{j}))^{2}\\ -(Q^{e}_{0}(a^{e}_{j}))^{-1}\Big\}+2(s_{1}-t_{1})\boldsymbol{\aleph}^{1}_{-1}(a^{e}_{j})\Big)\end{gathered}} & \boxed{\begin{gathered}\varkappa^{e}_{1}(a^{e}_{j})\varkappa^{e}_{2}(a^{e}_{j})\Big(s_{1}\Big\{Q^{e}_{1}(a^{e}_{j})\\ +Q^{e}_{0}(a^{e}_{j})\Big[\daleth^{1}_{1}(a^{e}_{j})+\daleth^{1}_{-1}(a^{e}_{j})\Big]\Big\}+t_{1}\\ \times\Big\{(Q^{e}_{0}(a^{e}_{j}))^{-1}+Q^{e}_{0}(a^{e}_{j})\boldsymbol{\aleph}^{1}_{1}(a^{e}_{j})\boldsymbol{\aleph}^{1}_{-1}(a^{e}_{j})\Big\}\\ +\mathrm{i}(s_{1}+t_{1})\Big\{\boldsymbol{\aleph}^{1}_{1}(a^{e}_{j})-\boldsymbol{\aleph}^{1}_{-1}(a^{e}_{j})\Big\}\Big)\end{gathered}} \end{pmatrix}, \tag{2.27}$$

$$s_{1}=\frac{5}{72}, \qquad t_{1}=-\frac{7}{72}, \qquad \mho^{e}_{i} := \begin{cases}\Omega^{e}_{i}, & i=1,\dots,N,\\ 0, & i=0,N+1,\end{cases} \tag{2.28}$$

$$Q^{e}_{0}(b^{e}_{0}) = -\mathrm{i}\left((a^{e}_{N+1}-b^{e}_{0})^{-1}\prod_{k=1}^{N}\left(\frac{b^{e}_{k}-b^{e}_{0}}{a^{e}_{k}-b^{e}_{0}}\right)\right)^{1/2}, \tag{2.29}$$

$$Q^{e}_{1}(b^{e}_{0}) = \frac{1}{2}Q^{e}_{0}(b^{e}_{0})\left(\sum_{k=1}^{N}\left(\frac{1}{b^{e}_{0}-b^{e}_{k}}-\frac{1}{b^{e}_{0}-a^{e}_{k}}\right)-\frac{1}{b^{e}_{0}-a^{e}_{N+1}}\right), \tag{2.30}$$

$$Q_0^e(a_{N+1}^e)=\left((a_{N+1}^e-b_0^e)\prod_{k=1}^{N}\left(\frac{a_{N+1}^e-b_k^e}{a_{N+1}^e-a_k^e}\right)\right)^{1/2}, \tag{2.31}$$

$$Q_1^e(a_{N+1}^e)=\frac{1}{2}Q_0^e(a_{N+1}^e)\left(\sum_{k=1}^{N}\left(\frac{1}{a_{N+1}^e-b_k^e}-\frac{1}{a_{N+1}^e-a_k^e}\right)+\frac{1}{a_{N+1}^e-b_0^e}\right), \tag{2.32}$$

$$Q_0^e(b_j^e)=-\mathrm{i}\left(\frac{(b_j^e-b_0^e)}{(a_{N+1}^e-b_j^e)(b_j^e-a_j^e)}\prod_{k=1}^{j-1}\left(\frac{b_j^e-b_k^e}{b_j^e-a_k^e}\right)\prod_{l=j+1}^{N}\left(\frac{b_l^e-b_j^e}{a_l^e-b_j^e}\right)\right)^{1/2}, \tag{2.33}$$

$$Q_1^e(b_j^e)=\frac{1}{2}Q_0^e(b_j^e)\left(\sum_{\substack{k=1\\k\neq j}}^{N}\left(\frac{1}{b_j^e-b_k^e}-\frac{1}{b_j^e-a_k^e}\right)+\frac{1}{b_j^e-b_0^e}-\frac{1}{b_j^e-a_{N+1}^e}-\frac{1}{b_j^e-a_j^e}\right), \tag{2.34}$$

$$Q_0^e(a_j^e)=\left(\frac{(a_j^e-b_0^e)(b_j^e-a_j^e)}{(a_{N+1}^e-a_j^e)}\prod_{k=1}^{j-1}\left(\frac{a_j^e-b_k^e}{a_j^e-a_k^e}\right)\prod_{l=j+1}^{N}\left(\frac{b_l^e-a_j^e}{a_l^e-a_j^e}\right)\right)^{1/2}, \tag{2.35}$$

$$Q_1^e(a_j^e)=\frac{1}{2}Q_0^e(a_j^e)\left(\sum_{\substack{k=1\\k\neq j}}^{N}\left(\frac{1}{a_j^e-b_k^e}-\frac{1}{a_j^e-a_k^e}\right)+\frac{1}{a_j^e-b_0^e}-\frac{1}{a_j^e-a_{N+1}^e}+\frac{1}{a_j^e-b_j^e}\right), \tag{2.36}$$

where $\mathrm{i}Q_0^e(b_{j-1}^e),\, Q_0^e(a_j^e)>0,\ j=1,\dots,N+1$,

$$\varkappa_1^e(\xi)=\frac{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\infty)+\boldsymbol{d}_e)\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\xi)-\frac{n}{2\pi}\boldsymbol{\Omega}^e+\boldsymbol{d}_e)}{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\infty)-\frac{n}{2\pi}\boldsymbol{\Omega}^e+\boldsymbol{d}_e)\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\xi)+\boldsymbol{d}_e)}, \tag{2.37}$$

$$\varkappa_2^e(\xi)=\frac{\boldsymbol{\theta}^e(-\boldsymbol{u}_+^e(\infty)-\boldsymbol{d}_e)\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\xi)-\frac{n}{2\pi}\boldsymbol{\Omega}^e-\boldsymbol{d}_e)}{\boldsymbol{\theta}^e(-\boldsymbol{u}_+^e(\infty)-\frac{n}{2\pi}\boldsymbol{\Omega}^e-\boldsymbol{d}_e)\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\xi)-\boldsymbol{d}_e)}, \tag{2.38}$$

$$\aleph_{\varepsilon_2}^{\varepsilon_1}(\xi)=-\frac{\mathfrak{u}^e(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi)}{\boldsymbol{\theta}^e(\varepsilon_1\boldsymbol{u}_+^e(\xi)+\varepsilon_2\boldsymbol{d}_e)}+\frac{\mathfrak{u}^e(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^e;\xi)}{\boldsymbol{\theta}^e(\varepsilon_1\boldsymbol{u}_+^e(\xi)-\frac{n}{2\pi}\boldsymbol{\Omega}^e+\varepsilon_2\boldsymbol{d}_e)},\quad \varepsilon_1,\varepsilon_2=\pm1, \tag{2.39}$$

$$\begin{aligned}\daleth_{\varepsilon_2}^{\varepsilon_1}(\xi)={}&-\frac{\mathfrak{v}^e(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi)}{\boldsymbol{\theta}^e(\varepsilon_1\boldsymbol{u}_+^e(\xi)+\varepsilon_2\boldsymbol{d}_e)}+\frac{\mathfrak{v}^e(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^e;\xi)}{\boldsymbol{\theta}^e(\varepsilon_1\boldsymbol{u}_+^e(\xi)-\frac{n}{2\pi}\boldsymbol{\Omega}^e+\varepsilon_2\boldsymbol{d}_e)}-\left(\frac{\mathfrak{u}^e(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi)}{\boldsymbol{\theta}^e(\varepsilon_1\boldsymbol{u}_+^e(\xi)+\varepsilon_2\boldsymbol{d}_e)}\right)^2\\&+\frac{\mathfrak{u}^e(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi)\mathfrak{u}^e(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^e;\xi)}{\boldsymbol{\theta}^e(\varepsilon_1\boldsymbol{u}_+^e(\xi)+\varepsilon_2\boldsymbol{d}_e)\boldsymbol{\theta}^e(\varepsilon_1\boldsymbol{u}_+^e(\xi)-\frac{n}{2\pi}\boldsymbol{\Omega}^e+\varepsilon_2\boldsymbol{d}_e)},\end{aligned} \tag{2.40}$$

$$\mathfrak{u}^e(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^e,\xi):=2\pi\Lambda_e^1(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^e,\xi),\qquad \mathfrak{v}^e(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^e,\xi):=-2\pi^2\Lambda_e^2(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^e,\xi), \tag{2.41}$$

$$\Lambda_e^{j'}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^e,\xi)=\sum_{m\in\mathbb{Z}^N}(\mathfrak{r}_e(\xi))^{j'}\mathrm{e}^{2\pi\mathrm{i}(m,\varepsilon_1\boldsymbol{u}_+^e(\xi)-\frac{n}{2\pi}\boldsymbol{\Omega}^e+\varepsilon_2\boldsymbol{d}_e)+\pi\mathrm{i}(m,\boldsymbol{\tau}^e m)},\quad j'=1,2, \tag{2.42}$$

$$\mathfrak{r}_e(\xi):=\frac{2(m,\vec{\perp}_e(\xi))}{\leftthreetimes^e(\xi)},\qquad \vec{\perp}_e(\xi)=(\measuredangle_1^e(\xi),\measuredangle_2^e(\xi),\dots,\measuredangle_N^e(\xi)), \tag{2.43}$$

$$\measuredangle_{j'}^e(\xi):=\sum_{k=1}^{N}c_{j'k}^e\xi^{N-k},\quad j'=1,\dots,N, \tag{2.44}$$

$$\leftthreetimes^e(b_0^e)=\mathrm{i}(-1)^N\eta_{b_0^e},\quad \leftthreetimes^e(a_{N+1}^e)=\eta_{a_{N+1}^e},\quad \leftthreetimes^e(b_j^e)=\mathrm{i}(-1)^{N-j}\eta_{b_j^e},\quad \leftthreetimes^e(a_j^e)=(-1)^{N-j+1}\eta_{a_j^e}, \tag{2.45}$$

$$\eta_{b_0^e}:=\left((a_{N+1}^e-b_0^e)\prod_{k=1}^{N}(b_k^e-b_0^e)(a_k^e-b_0^e)\right)^{1/2}, \tag{2.46}$$

$$\eta_{a_{N+1}^e}:=\left((a_{N+1}^e-b_0^e)\prod_{k=1}^{N}(a_{N+1}^e-b_k^e)(a_{N+1}^e-a_k^e)\right)^{1/2}, \tag{2.47}$$

$$\eta_{b_j^e}:=\left((b_j^e-a_j^e)(a_{N+1}^e-b_j^e)(b_j^e-b_0^e)\prod_{k=1}^{j-1}(b_j^e-b_k^e)(b_j^e-a_k^e)\prod_{l=j+1}^{N}(b_l^e-b_j^e)(a_l^e-b_j^e)\right)^{1/2}, \tag{2.48}$$

$$\eta_{a_j^e} := \left((b_j^e - a_j^e)(a_{N+1}^e - a_j^e)(a_j^e - b_0^e) \prod_{k=1}^{j-1} (a_j^e - b_k^e)(a_j^e - a_k^e) \prod_{l=j+1}^{N} (b_l^e - a_j^e)(a_l^e - a_j^e) \right)^{1/2}, \tag{2.49}$$

where $c_{j'k'}^e$, $j', k' = 1, \dots, N$, *are obtained from Equations* (E1) *and* (E2), $\eta_{b_{j-1}^e}, \eta_{a_j^e} > 0$, $j = 1, \dots, N+1$, *and*

$$\widehat{\alpha}_0^e(b_0^e) = \frac{4}{3}\mathrm{i}(-1)^N h_V^e(b_0^e)\eta_{b_0^e}, \tag{2.50}$$

$$\widehat{\alpha}_1^e(b_0^e) = \mathrm{i}(-1)^N \left(\frac{2}{5} h_V^e(b_0^e)\eta_{b_0^e} \left(\sum_{l=1}^{N} \left(\frac{1}{b_0^e - b_l^e} + \frac{1}{b_0^e - a_l^e} \right) + \frac{1}{b_0^e - a_{N+1}^e} \right) + \frac{4}{5}(h_V^e(b_0^e))' \eta_{b_0^e} \right), \tag{2.51}$$

$$\widehat{\alpha}_0^e(a_{N+1}^e) = \frac{4}{3} h_V^e(a_{N+1}^e)\eta_{a_{N+1}^e}, \tag{2.52}$$

$$\widehat{\alpha}_1^e(a_{N+1}^e) = \frac{2}{5} h_V^e(a_{N+1}^e)\eta_{a_{N+1}^e} \left(\sum_{l=1}^{N} \left(\frac{1}{a_{N+1}^e - b_l^e} + \frac{1}{a_{N+1}^e - a_l^e} \right) + \frac{1}{a_{N+1}^e - b_0^e} \right) + \frac{4}{5}(h_V^e(a_{N+1}^e))' \eta_{a_{N+1}^e}, \tag{2.53}$$

$$\widehat{\alpha}_0^e(b_j^e) = \frac{4}{3}\mathrm{i}(-1)^{N-j} h_V^e(b_j^e)\eta_{b_j^e}, \tag{2.54}$$

$$\widehat{\alpha}_1^e(b_j^e) = \mathrm{i}(-1)^{N-j} \left(\frac{2}{5} h_V^e(b_j^e)\eta_{b_j^e} \left(\sum_{\substack{k=1 \\ k \neq j}}^{N} \left(\frac{1}{b_j^e - b_k^e} + \frac{1}{b_j^e - a_k^e} \right) + \frac{1}{b_j^e - a_j^e} + \frac{1}{b_j^e - b_0^e} + \frac{1}{b_j^e - a_{N+1}^e} \right) + \frac{4}{5}(h_V^e(b_j^e))' \eta_{b_j^e} \right), \tag{2.55}$$

$$\widehat{\alpha}_0^e(a_j^e) = \frac{4}{3}(-1)^{N-j+1} h_V^e(a_j^e)\eta_{a_j^e}, \tag{2.56}$$

$$\widehat{\alpha}_1^e(a_j^e) = (-1)^{N-j+1} \left(\frac{2}{5} h_V^e(a_j^e)\eta_{a_j^e} \left(\sum_{\substack{k=1 \\ k \neq j}}^{N} \left(\frac{1}{a_j^e - b_k^e} + \frac{1}{a_j^e - a_k^e} \right) + \frac{1}{a_j^e - b_j^e} + \frac{1}{a_j^e - a_{N+1}^e} + \frac{1}{a_j^e - b_0^e} \right) + \frac{4}{5}(h_V^e(a_j^e))' \eta_{a_j^e} \right); \tag{2.57}$$

(2) *for* $z \in \Upsilon_2^e$ $(\subset \mathbb{C}_-)$,

$$\boldsymbol{\pi}_{2n}(z) \underset{n\to\infty}{=} \exp(n(g^e(z) + Q_e)) \left((\overset{e}{m}{}^{\infty}(z))_{11} \left(1 + \frac{1}{n} (\mathcal{R}_\infty^e(z))_{11} + \mathcal{O}\left(\frac{1}{n^2} \right) \right) + (\overset{e}{m}{}^{\infty}(z))_{21} \left(\frac{1}{n} (\mathcal{R}_\infty^e(z))_{12} + \mathcal{O}\left(\frac{1}{n^2} \right) \right) \right), \tag{2.58}$$

and

$$\int_{\mathbb{R}} \frac{\boldsymbol{\pi}_{2n}(s)\mathrm{e}^{-n\widetilde{V}(s)}}{s-z} \frac{\mathrm{d}s}{2\pi\mathrm{i}} \underset{n\to\infty}{=} \exp(-n(g^e(z) - \ell_e + Q_e)) \left((\overset{e}{m}{}^{\infty}(z))_{12} \left(1 + \frac{1}{n} (\mathcal{R}_\infty^e(z))_{11} + \mathcal{O}\left(\frac{1}{n^2} \right) \right) + (\overset{e}{m}{}^{\infty}(z))_{22} \left(\frac{1}{n} (\mathcal{R}_\infty^e(z))_{12} + \mathcal{O}\left(\frac{1}{n^2} \right) \right) \right); \tag{2.59}$$

(3) *for* $z \in \Upsilon_3^e$ $(\subset \cup_{j=1}^{N+1} \left\{ z \in \mathbb{C}^*;\ \mathrm{Re}(z) \in (b_{j-1}^e, a_j^e),\ \inf_{q \in (b_{j-1}^e, a_j^e)} |z-q| < 2^{-1/2} \min\{\delta_{b_{j-1}}^e, \delta_{a_j}^e\} \right\} \subset \mathbb{C}_+)$,

$$\boldsymbol{\pi}_{2n}(z) \underset{n\to\infty}{=} \exp(n(g^e(z) + Q_e)) \left(\left((\overset{e}{m}{}^{\infty}(z))_{11} + (\overset{e}{m}{}^{\infty}(z))_{12} \mathrm{e}^{-4n\pi\mathrm{i} \int_z^{a_{N+1}^e} \psi_V^e(s)\,\mathrm{d}s} \right) \left(1 + \frac{1}{n} (\mathcal{R}_\infty^e(z))_{11} + \mathcal{O}\left(\frac{1}{n^2} \right) \right) + \left((\overset{e}{m}{}^{\infty}(z))_{21} + (\overset{e}{m}{}^{\infty}(z))_{22} \mathrm{e}^{-4n\pi\mathrm{i} \int_z^{a_{N+1}^e} \psi_V^e(s)\,\mathrm{d}s} \right) \left(\frac{1}{n} (\mathcal{R}_\infty^e(z))_{12} + \mathcal{O}\left(\frac{1}{n^2} \right) \right) \right), \tag{2.60}$$

and

$$\int_{\mathbb{R}} \frac{\boldsymbol{\pi}_{2n}(s)\mathrm{e}^{-n\widetilde{V}(s)}}{s-z} \frac{\mathrm{d}s}{2\pi\mathrm{i}} \underset{n\to\infty}{=} \exp(-n(g^e(z) - \ell_e + Q_e)) \left((\overset{e}{m}{}^{\infty}(z))_{12} \left(1 + \frac{1}{n} (\mathcal{R}_\infty^e(z))_{11} + \mathcal{O}\left(\frac{1}{n^2} \right) \right) \right.$$

$$+ \ (\overset{e}{m}^{\infty}(z))_{22}\left(\frac{1}{n}(\mathcal{R}^{e}_{\infty}(z))_{12}+\mathcal{O}\left(\frac{1}{n^{2}}\right)\right)\Big); \tag{2.61}$$

(4) *for* $z\in\Upsilon^{e}_{4}$ $(\subset\cup_{j=1}^{N+1}\left\{z\in\mathbb{C}^{\ast};\ \mathrm{Re}(z)\in(b^{e}_{j-1},a^{e}_{j}),\ \inf_{q\in(b^{e}_{j-1},a^{e}_{j})}|z-q|<2^{-1/2}\min\{\delta^{e}_{b_{j-1}},\delta^{e}_{a_{j}}\}\right\}\subset\mathbb{C}_{-})$,

$$\begin{aligned}\boldsymbol{\pi}_{2n}(z) \underset{n\to\infty}{=} &\exp(n(g^{e}(z)+Q_{e}))\Big(\Big((\overset{e}{m}^{\infty}(z))_{11}-(\overset{e}{m}^{\infty}(z))_{12}\mathrm{e}^{4n\pi\mathrm{i}\int_{z}^{a^{e}_{N+1}}\psi^{e}_{V}(s)\,\mathrm{d}s}\Big)\Big(1+\frac{1}{n}(\mathcal{R}^{e}_{\infty}(z))_{11}\\ &+\mathcal{O}\left(\frac{1}{n^{2}}\right)\Big)+\Big((\overset{e}{m}^{\infty}(z))_{21}-(\overset{e}{m}^{\infty}(z))_{22}\mathrm{e}^{4n\pi\mathrm{i}\int_{z}^{a^{e}_{N+1}}\psi^{e}_{V}(s)\,\mathrm{d}s}\Big)\Big(\frac{1}{n}(\mathcal{R}^{e}_{\infty}(z))_{12}+\mathcal{O}\left(\frac{1}{n^{2}}\right)\Big)\Big),\end{aligned} \tag{2.62}$$

and

$$\begin{aligned}\int_{\mathbb{R}}\frac{\boldsymbol{\pi}_{2n}(s)\mathrm{e}^{-n\widetilde{V}(s)}}{s-z}\,\frac{\mathrm{d}s}{2\pi\mathrm{i}} \underset{n\to\infty}{=} &\exp(-n(g^{e}(z)-\ell_{e}+Q_{e}))\Big((\overset{e}{m}^{\infty}(z))_{12}\Big(1+\frac{1}{n}(\mathcal{R}^{e}_{\infty}(z))_{11}+\mathcal{O}\left(\frac{1}{n^{2}}\right)\Big)\\ &+\ (\overset{e}{m}^{\infty}(z))_{22}\left(\frac{1}{n}(\mathcal{R}^{e}_{\infty}(z))_{12}+\mathcal{O}\left(\frac{1}{n^{2}}\right)\right)\Big);\end{aligned} \tag{2.63}$$

(5) *for* $z\in\Omega^{e,1}_{b^{e}_{j-1}}$ $(\subset\mathbb{C}_{+}\cap\mathbb{U}^{e}_{\delta_{b_{j-1}}})$, $j=1,\dots,N+1$,

$$\begin{aligned}\boldsymbol{\pi}_{2n}(z) \underset{n\to\infty}{=} &\exp(n(g^{e}(z)+Q_{e}))\Big((m^{b,1}_{p}(z))_{11}\Big(1+\frac{1}{n}\Big(\mathcal{R}^{e}_{\infty}(z)-\widetilde{\mathcal{R}}^{e}_{\infty}(z)\Big)_{11}+\mathcal{O}\left(\frac{1}{n^{2}}\right)\Big)\\ &+\ (m^{b,1}_{p}(z))_{21}\Big(\frac{1}{n}\Big(\mathcal{R}^{e}_{\infty}(z)-\widetilde{\mathcal{R}}^{e}_{\infty}(z)\Big)_{12}+\mathcal{O}\left(\frac{1}{n^{2}}\right)\Big)\Big),\end{aligned} \tag{2.64}$$

and

$$\begin{aligned}\int_{\mathbb{R}}\frac{\boldsymbol{\pi}_{2n}(s)\mathrm{e}^{-n\widetilde{V}(s)}}{s-z}\,\frac{\mathrm{d}s}{2\pi\mathrm{i}} \underset{n\to\infty}{=} &\exp(-n(g^{e}(z)-\ell_{e}+Q_{e}))\Big((m^{b,1}_{p}(z))_{12}\Big(1+\frac{1}{n}\Big(\mathcal{R}^{e}_{\infty}(z)-\widetilde{\mathcal{R}}^{e}_{\infty}(z)\Big)_{11}\\ &+\ \mathcal{O}\left(\frac{1}{n^{2}}\right)\Big)+(m^{b,1}_{p}(z))_{22}\Big(\frac{1}{n}\Big(\mathcal{R}^{e}_{\infty}(z)-\widetilde{\mathcal{R}}^{e}_{\infty}(z)\Big)_{12}+\mathcal{O}\left(\frac{1}{n^{2}}\right)\Big)\Big),\end{aligned} \tag{2.65}$$

where

$$\begin{aligned}(m^{b,1}_{p}(z))_{11} :=& -\mathrm{i}\sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}n\xi^{e}_{b_{j-1}}(z)}\Big(\mathrm{i}\Big(\mathrm{Ai}(p_{b})(p_{b})^{1/4}-\mathrm{Ai}'(p_{b})(p_{b})^{-1/4}\Big)(\overset{e}{m}^{\infty}(z))_{11}\\ &-\Big(\mathrm{Ai}(p_{b})(p_{b})^{1/4}+\mathrm{Ai}'(p_{b})(p_{b})^{-1/4}\Big)(\overset{e}{m}^{\infty}(z))_{12}\mathrm{e}^{-\mathrm{i}n\mho^{e}_{j-1}}\Big),\end{aligned} \tag{2.66}$$

$$\begin{aligned}(m^{b,1}_{p}(z))_{12} :=& \sqrt{\pi}\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}}\mathrm{e}^{-\frac{1}{2}n\xi^{e}_{b_{j-1}}(z)}\Big(\mathrm{i}\Big(-\mathrm{Ai}(\omega^{2}p_{b})(p_{b})^{1/4}+\omega^{2}\,\mathrm{Ai}'(\omega^{2}p_{b})(p_{b})^{-1/4}\Big)(\overset{e}{m}^{\infty}(z))_{11}\\ &\times\mathrm{e}^{\mathrm{i}n\mho^{e}_{j-1}}+\Big(\mathrm{Ai}(\omega^{2}p_{b})(p_{b})^{1/4}+\omega^{2}\,\mathrm{Ai}'(\omega^{2}p_{b})(p_{b})^{-1/4}\Big)(\overset{e}{m}^{\infty}(z))_{12}\Big),\end{aligned} \tag{2.67}$$

$$\begin{aligned}(m^{b,1}_{p}(z))_{21} :=& -\mathrm{i}\sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}n\xi^{e}_{b_{j-1}}(z)}\Big(\mathrm{i}\Big(\mathrm{Ai}(p_{b})(p_{b})^{1/4}-\mathrm{Ai}'(p_{b})(p_{b})^{-1/4}\Big)(\overset{e}{m}^{\infty}(z))_{21}\\ &-\Big(\mathrm{Ai}(p_{b})(p_{b})^{1/4}+\mathrm{Ai}'(p_{b})(p_{b})^{-1/4}\Big)(\overset{e}{m}^{\infty}(z))_{22}\mathrm{e}^{-\mathrm{i}n\mho^{e}_{j-1}}\Big),\end{aligned} \tag{2.68}$$

$$\begin{aligned}(m^{b,1}_{p}(z))_{22} :=& \sqrt{\pi}\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}}\mathrm{e}^{-\frac{1}{2}n\xi^{e}_{b_{j-1}}(z)}\Big(\mathrm{i}\Big(-\mathrm{Ai}(\omega^{2}p_{b})(p_{b})^{1/4}+\omega^{2}\,\mathrm{Ai}'(\omega^{2}p_{b})(p_{b})^{-1/4}\Big)(\overset{e}{m}^{\infty}(z))_{21}\\ &\times\mathrm{e}^{\mathrm{i}n\mho^{e}_{j-1}}+\Big(\mathrm{Ai}(\omega^{2}p_{b})(p_{b})^{1/4}+\omega^{2}\,\mathrm{Ai}'(\omega^{2}p_{b})(p_{b})^{-1/4}\Big)(\overset{e}{m}^{\infty}(z))_{22}\Big),\end{aligned} \tag{2.69}$$

with $\omega=\exp(2\pi\mathrm{i}/3)$,

$$\widetilde{\mathcal{R}}^{e}_{\infty}(z):=\sum_{j=1}^{N+1}\Big(\mathcal{R}^{\infty}_{b^{e}_{j-1}}(z)\mathbf{1}_{\mathbb{U}^{e}_{\delta_{b_{j-1}}}}(z)+\mathcal{R}^{\infty}_{a^{e}_{j}}(z)\mathbf{1}_{\mathbb{U}^{e}_{\delta_{a_{j}}}}(z)\Big), \tag{2.70}$$

$$\xi^{e}_{b_{j-1}}(z)=-2\int_{z}^{b^{e}_{j-1}}(R_{e}(s))^{1/2}h^{e}_{V}(s)\,\mathrm{d}s, \qquad p_{b}=\Big(\frac{3}{4}n\xi^{e}_{b_{j-1}}(z)\Big)^{2/3}, \tag{2.71}$$

$$\mathcal{R}^{\infty}_{b^{e}_{j-1}}(z)=\frac{1}{\xi^{e}_{b_{j-1}}(z)}\overset{e}{m}^{\infty}(z)\begin{pmatrix}-(s_{1}+t_{1}) & -\mathrm{i}(s_{1}-t_{1})\mathrm{e}^{\mathrm{i}n\mho^{e}_{j-1}}\\ -\mathrm{i}(s_{1}-t_{1})\mathrm{e}^{-\mathrm{i}n\mho^{e}_{j-1}} & (s_{1}+t_{1})\end{pmatrix}(\overset{e}{m}^{\infty}(z))^{-1}, \tag{2.72}$$

$$\mathcal{R}^{\infty}_{a_j^e}(z) = \frac{1}{\xi^e_{a_j}(z)} \overset{e}{m}{}^{\infty}(z) \begin{pmatrix} -(s_1+t_1) & \mathrm{i}(s_1-t_1)\mathrm{e}^{\mathrm{i}n\mho^e_j} \\ \mathrm{i}(s_1-t_1)\mathrm{e}^{-\mathrm{i}n\mho^e_j} & (s_1+t_1) \end{pmatrix} (\overset{e}{m}{}^{\infty}(z))^{-1}, \tag{2.73}$$

$$\xi^e_{a_j}(z) = 2\int_{a_j^e}^{z} (R_e(s))^{1/2} h^e_V(s)\,\mathrm{d}s, \tag{2.74}$$

and $\mathbf{1}_{\mathbb{U}^e_{\delta_{b_{j-1}}}}(z)$ *(resp.,* $\mathbf{1}_{\mathbb{U}^e_{\delta_{a_j}}}(z)$*) the indicator (characteristic) function of the set* $\mathbb{U}^e_{\delta_{b_{j-1}}}$ *(resp.,* $\mathbb{U}^e_{\delta_{a_j}}$*);*
(6) *for* $z \in \Omega^{e,2}_{b_{j-1}}$ $(\subset \mathbb{C}_+ \cap \mathbb{U}^e_{\delta_{b_{j-1}}})$, $j=1,\dots,N+1$,

$$\begin{aligned} \boldsymbol{\pi}_{2n}(z) \underset{n\to\infty}{=} & \exp(n(g^e(z)+Q_e))\Bigg(\Big((m^{b,2}_p(z))_{11} + (m^{b,2}_p(z))_{12}\mathrm{e}^{-4n\pi\mathrm{i}\int_z^{a^e_{N+1}}\psi^e_V(s)\,\mathrm{d}s}\Big)\Big(1+\frac{1}{n}(\mathcal{R}^e_\infty(z) \\ & -\widetilde{\mathcal{R}}^e_\infty(z)\Big)_{11} + O\Big(\frac{1}{n^2}\Big)\Big) + \Big((m^{b,2}_p(z))_{21} + (m^{b,2}_p(z))_{22}\mathrm{e}^{-4n\pi\mathrm{i}\int_z^{a^e_{N+1}}\psi^e_V(s)\,\mathrm{d}s}\Big)\Big(\frac{1}{n}(\mathcal{R}^e_\infty(z) \\ & -\widetilde{\mathcal{R}}^e_\infty(z)\Big)_{12} + O\Big(\frac{1}{n^2}\Big)\Big)\Bigg), \end{aligned} \tag{2.75}$$

and

$$\begin{aligned} \int_{\mathbb{R}} \frac{\boldsymbol{\pi}_{2n}(s)\mathrm{e}^{-n\widetilde{V}(s)}}{s-z}\frac{\mathrm{d}s}{2\pi\mathrm{i}} \underset{n\to\infty}{=} & \exp(-n(g^e(z)-\ell_e+Q_e))\Big((m^{b,2}_p(z))_{12}\Big(1+\frac{1}{n}\Big(\mathcal{R}^e_\infty(z)-\widetilde{\mathcal{R}}^e_\infty(z)\Big)_{11} \\ & + O\Big(\frac{1}{n^2}\Big)\Big) + (m^{b,2}_p(z))_{22}\Big(\frac{1}{n}\Big(\mathcal{R}^e_\infty(z)-\widetilde{\mathcal{R}}^e_\infty(z)\Big)_{12} + O\Big(\frac{1}{n^2}\Big)\Big)\Big), \end{aligned} \tag{2.76}$$

where

$$\begin{aligned} (m^{b,2}_p(z))_{11} :=& -\mathrm{i}\sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}n\xi^e_{b_{j-1}}(z)}\Big(\mathrm{i}\Big(-\omega\,\mathrm{Ai}(\omega p_b)(p_b)^{1/4} + \omega^2\,\mathrm{Ai}'(\omega p_b)(p_b)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{11} \\ & + \Big(\omega\,\mathrm{Ai}(\omega p_b)(p_b)^{1/4} + \omega^2\,\mathrm{Ai}'(\omega p_b)(p_b)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{12}\mathrm{e}^{-\mathrm{i}n\mho^e_{j-1}}\Big), \end{aligned} \tag{2.77}$$

$$\begin{aligned} (m^{b,2}_p(z))_{12} :=& \sqrt{\pi}\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}}\mathrm{e}^{-\frac{1}{2}n\xi^e_{b_{j-1}}(z)}\Big(\mathrm{i}\Big(-\mathrm{Ai}(\omega^2 p_b)(p_b)^{1/4} + \omega^2\,\mathrm{Ai}'(\omega p_b)(p_b)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{11} \\ & \times \mathrm{e}^{\mathrm{i}n\mho^e_{j-1}} + \Big(\mathrm{Ai}(\omega^2 p_b)(p_b)^{1/4} + \omega^2\,\mathrm{Ai}'(\omega p_b)(p_b)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{12}\Big), \end{aligned} \tag{2.78}$$

$$\begin{aligned} (m^{b,2}_p(z))_{21} :=& -\mathrm{i}\sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}n\xi^e_{b_{j-1}}(z)}\Big(\mathrm{i}\Big(-\omega\,\mathrm{Ai}(\omega p_b)(p_b)^{1/4} + \omega^2\,\mathrm{Ai}'(\omega p_b)(p_b)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{21} \\ & + \Big(\omega\,\mathrm{Ai}(\omega p_b)(p_b)^{1/4} + \omega^2\,\mathrm{Ai}'(\omega p_b)(p_b)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{22}\mathrm{e}^{-\mathrm{i}n\mho^e_{j-1}}\Big), \end{aligned} \tag{2.79}$$

$$\begin{aligned} (m^{b,2}_p(z))_{22} :=& \sqrt{\pi}\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}}\mathrm{e}^{-\frac{1}{2}n\xi^e_{b_{j-1}}(z)}\Big(\mathrm{i}\Big(-\mathrm{Ai}(\omega^2 p_b)(p_b)^{1/4} + \omega^2\,\mathrm{Ai}'(\omega p_b)(p_b)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{21} \\ & \times \mathrm{e}^{\mathrm{i}n\mho^e_{j-1}} + \Big(\mathrm{Ai}(\omega^2 p_b)(p_b)^{1/4} + \omega^2\,\mathrm{Ai}'(\omega p_b)(p_b)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{22}\Big); \end{aligned} \tag{2.80}$$

(7) *for* $z \in \Omega^{e,3}_{b_{j-1}}$ $(\subset \mathbb{C}_- \cap \mathbb{U}^e_{\delta_{b_{j-1}}})$, $j=1,\dots,N+1$,

$$\begin{aligned} \boldsymbol{\pi}_{2n}(z) \underset{n\to\infty}{=} & \exp(n(g^e(z)+Q_e))\Bigg(\Big((m^{b,3}_p(z))_{11} - (m^{b,3}_p(z))_{12}\mathrm{e}^{4n\pi\mathrm{i}\int_z^{a^e_{N+1}}\psi^e_V(s)\,\mathrm{d}s}\Big)\Big(1+\frac{1}{n}(\mathcal{R}^e_\infty(z) \\ & -\widetilde{\mathcal{R}}^e_\infty(z)\Big)_{11} + O\Big(\frac{1}{n^2}\Big)\Big) + \Big((m^{b,3}_p(z))_{21} - (m^{b,3}_p(z))_{22}\mathrm{e}^{4n\pi\mathrm{i}\int_z^{a^e_{N+1}}\psi^e_V(s)\,\mathrm{d}s}\Big)\Big(\frac{1}{n}(\mathcal{R}^e_\infty(z) \\ & -\widetilde{\mathcal{R}}^e_\infty(z)\Big)_{12} + O\Big(\frac{1}{n^2}\Big)\Big)\Bigg), \end{aligned} \tag{2.81}$$

and

$$\begin{aligned} \int_{\mathbb{R}} \frac{\boldsymbol{\pi}_{2n}(s)\mathrm{e}^{-n\widetilde{V}(s)}}{s-z}\frac{\mathrm{d}s}{2\pi\mathrm{i}} \underset{n\to\infty}{=} & \exp(-n(g^e(z)-\ell_e+Q_e))\Big((m^{b,3}_p(z))_{12}\Big(1+\frac{1}{n}\Big(\mathcal{R}^e_\infty(z)-\widetilde{\mathcal{R}}^e_\infty(z)\Big)_{11} \\ & + O\Big(\frac{1}{n^2}\Big)\Big) + (m^{b,3}_p(z))_{22}\Big(\frac{1}{n}\Big(\mathcal{R}^e_\infty(z)-\widetilde{\mathcal{R}}^e_\infty(z)\Big)_{12} + O\Big(\frac{1}{n^2}\Big)\Big)\Big), \end{aligned} \tag{2.82}$$

where

$$(m^{b,3}_p(z))_{11} := -\mathrm{i}\sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}n\xi^e_{b_{j-1}}(z)}\Big(\mathrm{i}\Big(-\omega^2\,\mathrm{Ai}(\omega^2 p_b)(p_b)^{1/4} + \omega\,\mathrm{Ai}'(\omega^2 p_b)(p_b)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{11}$$

$$+\Big(\omega^2\,\mathrm{Ai}(\omega^2 p_b)(p_b)^{1/4}+\omega\,\mathrm{Ai}'(\omega^2 p_b)(p_b)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{12}\mathrm{e}^{\mathrm{i}n\mho^e_{j-1}}\Big), \tag{2.83}$$

$$(m_p^{b,3}(z))_{12} := \sqrt{\pi}\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}}\mathrm{e}^{-\frac{1}{2}n\xi^e_{b_{j-1}}(z)}\Big(\mathrm{i}\Big(\omega^2\,\mathrm{Ai}(\omega p_b)(p_b)^{1/4}-\mathrm{Ai}'(\omega p_b)(p_b)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{11}$$
$$\times\,\mathrm{e}^{-\mathrm{i}n\mho^e_{j-1}}-\Big(\omega^2\,\mathrm{Ai}(\omega p_b)(p_b)^{1/4}+\mathrm{Ai}'(\omega p_b)(p_b)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{12}\Big), \tag{2.84}$$

$$(m_p^{b,3}(z))_{21} := -\,\mathrm{i}\sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}n\xi^e_{b_{j-1}}(z)}\Big(\mathrm{i}\Big(-\omega^2\,\mathrm{Ai}(\omega^2 p_b)(p_b)^{1/4}+\omega\,\mathrm{Ai}'(\omega^2 p_b)(p_b)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{21}$$
$$+\Big(\omega^2\,\mathrm{Ai}(\omega^2 p_b)(p_b)^{1/4}+\omega\,\mathrm{Ai}'(\omega^2 p_b)(p_b)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{22}\mathrm{e}^{\mathrm{i}n\mho^e_{j-1}}\Big), \tag{2.85}$$

$$(m_p^{b,3}(z))_{22} := \sqrt{\pi}\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}}\mathrm{e}^{-\frac{1}{2}n\xi^e_{b_{j-1}}(z)}\Big(\mathrm{i}\Big(\omega^2\,\mathrm{Ai}(\omega p_b)(p_b)^{1/4}-\mathrm{Ai}'(\omega p_b)(p_b)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{21}$$
$$\times\,\mathrm{e}^{-\mathrm{i}n\mho^e_{j-1}}-\Big(\omega^2\,\mathrm{Ai}(\omega p_b)(p_b)^{1/4}+\mathrm{Ai}'(\omega p_b)(p_b)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{22}\Big); \tag{2.86}$$

(8) *for* $z\in\Omega^{e,4}_{b_{j-1}}$ $(\subset\mathbb{C}_-\cap\mathbb{U}^e_{\delta_{b_{j-1}}})$, $j=1,\dots,N+1$,

$$\boldsymbol{\pi}_{2n}(z)\underset{n\to\infty}{=}\exp(n(g^e(z)+Q_e))\Big((m_p^{b,4}(z))_{11}\Big(1+\frac{1}{n}\Big(\mathcal{R}^e_\infty(z)-\widetilde{\mathcal{R}}^e_\infty(z)\Big)_{11}+O\Big(\frac{1}{n^2}\Big)\Big)$$
$$+\;(m_p^{b,4}(z))_{21}\Big(\frac{1}{n}\Big(\mathcal{R}^e_\infty(z)-\widetilde{\mathcal{R}}^e_\infty(z)\Big)_{12}+O\Big(\frac{1}{n^2}\Big)\Big)\Big), \tag{2.87}$$

and

$$\int_{\mathbb{R}}\frac{\boldsymbol{\pi}_{2n}(s)\mathrm{e}^{-n\widetilde{V}(s)}}{s-z}\,\frac{\mathrm{d}s}{2\pi\mathrm{i}}\underset{n\to\infty}{=}\exp(-n(g^e(z)-\ell_e+Q_e))\Big((m_p^{b,4}(z))_{12}\Big(1+\frac{1}{n}\Big(\mathcal{R}^e_\infty(z)-\widetilde{\mathcal{R}}^e_\infty(z)\Big)_{11}$$
$$+\;O\Big(\frac{1}{n^2}\Big)\Big)+(m_p^{b,4}(z))_{22}\Big(\frac{1}{n}\Big(\mathcal{R}^e_\infty(z)-\widetilde{\mathcal{R}}^e_\infty(z)\Big)_{12}+O\Big(\frac{1}{n^2}\Big)\Big)\Big), \tag{2.88}$$

where

$$(m_p^{b,4}(z))_{11} := -\,\mathrm{i}\sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}n\xi^e_{b_{j-1}}(z)}\Big(\mathrm{i}\Big(\mathrm{Ai}(p_b)(p_b)^{1/4}-\mathrm{Ai}'(p_b)(p_b)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{11}$$
$$-\Big(\mathrm{Ai}(p_b)(p_b)^{1/4}+\mathrm{Ai}'(p_b)(p_b)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{12}\mathrm{e}^{\mathrm{i}n\mho^e_{j-1}}\Big), \tag{2.89}$$

$$(m_p^{b,4}(z))_{12} := \sqrt{\pi}\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}}\mathrm{e}^{-\frac{1}{2}n\xi^e_{b_{j-1}}(z)}\Big(\mathrm{i}\Big(\omega^2\,\mathrm{Ai}(\omega p_b)(p_b)^{1/4}-\mathrm{Ai}'(\omega p_b)(p_b)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{11}$$
$$\times\,\mathrm{e}^{-\mathrm{i}n\mho^e_{j-1}}-\Big(\omega^2\,\mathrm{Ai}(\omega p_b)(p_b)^{1/4}+\mathrm{Ai}'(\omega p_b)(p_b)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{12}\Big), \tag{2.90}$$

$$(m_p^{b,4}(z))_{21} := -\,\mathrm{i}\sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}n\xi^e_{b_{j-1}}(z)}\Big(\mathrm{i}\Big(\mathrm{Ai}(p_b)(p_b)^{1/4}-\mathrm{Ai}'(p_b)(p_b)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{21}$$
$$-\Big(\mathrm{Ai}(p_b)(p_b)^{1/4}+\mathrm{Ai}'(p_b)(p_b)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{22}\mathrm{e}^{\mathrm{i}n\mho^e_{j-1}}\Big), \tag{2.91}$$

$$(m_p^{b,4}(z))_{22} := \sqrt{\pi}\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}}\mathrm{e}^{-\frac{1}{2}n\xi^e_{b_{j-1}}(z)}\Big(\mathrm{i}\Big(\omega^2\,\mathrm{Ai}(\omega p_b)(p_b)^{1/4}-\mathrm{Ai}'(\omega p_b)(p_b)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{21}$$
$$\times\,\mathrm{e}^{-\mathrm{i}n\mho^e_{j-1}}-\Big(\omega^2\,\mathrm{Ai}(\omega p_b)(p_b)^{1/4}+\mathrm{Ai}'(\omega p_b)(p_b)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{22}\Big); \tag{2.92}$$

(9) *for* $z\in\Omega^{e,1}_{a_j}$ $(\subset\mathbb{C}_+\cap\mathbb{U}^e_{\delta_{a_j}})$, $j=1,\dots,N+1$,

$$\boldsymbol{\pi}_{2n}(z)\underset{n\to\infty}{=}\exp(n(g^e(z)+Q_e))\Big((m_p^{a,1}(z))_{11}\Big(1+\frac{1}{n}\Big(\mathcal{R}^e_\infty(z)-\widetilde{\mathcal{R}}^e_\infty(z)\Big)_{11}+O\Big(\frac{1}{n^2}\Big)\Big)$$
$$+\;(m_p^{a,1}(z))_{21}\Big(\frac{1}{n}\Big(\mathcal{R}^e_\infty(z)-\widetilde{\mathcal{R}}^e_\infty(z)\Big)_{12}+O\Big(\frac{1}{n^2}\Big)\Big)\Big), \tag{2.93}$$

and

$$\int_{\mathbb{R}}\frac{\boldsymbol{\pi}_{2n}(s)\mathrm{e}^{-n\widetilde{V}(s)}}{s-z}\,\frac{\mathrm{d}s}{2\pi\mathrm{i}}\underset{n\to\infty}{=}\exp(-n(g^e(z)-\ell_e+Q_e))\Big((m_p^{a,1}(z))_{12}\Big(1+\frac{1}{n}\Big(\mathcal{R}^e_\infty(z)-\widetilde{\mathcal{R}}^e_\infty(z)\Big)_{11}$$
$$+\;O\Big(\frac{1}{n^2}\Big)\Big)+(m_p^{a,1}(z))_{22}\Big(\frac{1}{n}\Big(\mathcal{R}^e_\infty(z)-\widetilde{\mathcal{R}}^e_\infty(z)\Big)_{12}+O\Big(\frac{1}{n^2}\Big)\Big)\Big), \tag{2.94}$$

where

$$(m_p^{a,1}(z))_{11} := -\,\mathrm{i}\sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}n\xi^e_{a_j}(z)}\Big(\mathrm{i}\Big(\mathrm{Ai}(p_a)(p_a)^{1/4}-\mathrm{Ai}'(p_a)(p_a)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{11}$$

$$+\Big(\mathrm{Ai}(p_a)(p_a)^{1/4}+\mathrm{Ai}'(p_a)(p_a)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{12}\mathrm{e}^{-\mathrm{i}n\mho_j^e}\Big), \tag{2.95}$$

$$(m_p^{a,1}(z))_{12} := \sqrt{\pi}\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}}\mathrm{e}^{-\frac{1}{2}n\xi_{a_j}^e(z)}\Big(\mathrm{i}\Big(\mathrm{Ai}(\omega^2p_a)(p_a)^{1/4}-\omega^2\,\mathrm{Ai}'(\omega^2p_a)(p_a)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{11}$$
$$\times\,\mathrm{e}^{\mathrm{i}n\mho_j^e}+\Big(\mathrm{Ai}(\omega^2p_a)(p_a)^{1/4}+\omega^2\,\mathrm{Ai}'(\omega^2p_a)(p_a)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{12}\Big), \tag{2.96}$$

$$(m_p^{a,1}(z))_{21} := -\mathrm{i}\,\sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}n\xi_{a_j}^e(z)}\Big(\mathrm{i}\Big(\mathrm{Ai}(p_a)(p_a)^{1/4}-\mathrm{Ai}'(p_a)(p_a)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{21}$$
$$+\Big(\mathrm{Ai}(p_a)(p_a)^{1/4}+\mathrm{Ai}'(p_a)(p_a)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{22}\mathrm{e}^{-\mathrm{i}n\mho_j^e}\Big), \tag{2.97}$$

$$(m_p^{a,1}(z))_{22} := \sqrt{\pi}\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}}\mathrm{e}^{-\frac{1}{2}n\xi_{a_j}^e(z)}\Big(\mathrm{i}\Big(\mathrm{Ai}(\omega^2p_a)(p_a)^{1/4}-\omega^2\,\mathrm{Ai}'(\omega^2p_a)(p_a)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{21}$$
$$\times\,\mathrm{e}^{\mathrm{i}n\mho_j^e}+\Big(\mathrm{Ai}(\omega^2p_a)(p_a)^{1/4}+\omega^2\,\mathrm{Ai}'(\omega^2p_a)(p_a)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{22}\Big), \tag{2.98}$$

with

$$p_a=\Big(\frac{3}{4}n\xi_{a_j}^e(z)\Big)^{2/3}; \tag{2.99}$$

(10) *for* $z\in\Omega_{a_j}^{e,2}$ $(\subset\mathbb{C}_+\cap\mathbb{U}_{\delta_{a_j}}^e)$, $j=1,\dots,N+1$,

$$\boldsymbol{\pi}_{2n}(z) \underset{n\to\infty}{=} \exp(n(g^e(z)+Q_e))\bigg(\Big((m_p^{a,2}(z))_{11}+(m_p^{a,2}(z))_{12}\mathrm{e}^{-4n\pi\mathrm{i}\int_z^{a_{N+1}^e}\psi_V^e(s)\,\mathrm{d}s}\Big)\Big(1+\frac{1}{n}(\mathcal{R}_{\infty}^e(z)$$
$$-\ \widetilde{\mathcal{R}}_{\infty}^e(z)\Big)_{11}+\mathcal{O}\Big(\frac{1}{n^2}\Big)\Big)+\Big((m_p^{a,2}(z))_{21}+(m_p^{a,2}(z))_{22}\mathrm{e}^{-4n\pi\mathrm{i}\int_z^{a_{N+1}^e}\psi_V^e(s)\,\mathrm{d}s}\Big)\Big(\frac{1}{n}(\mathcal{R}_{\infty}^e(z)$$
$$-\ \widetilde{\mathcal{R}}_{\infty}^e(z)\Big)_{12}+\mathcal{O}\Big(\frac{1}{n^2}\Big)\Big)\bigg), \tag{2.100}$$

and

$$\int_{\mathbb{R}}\frac{\boldsymbol{\pi}_{2n}(s)\mathrm{e}^{-n\widetilde{V}(s)}}{s-z}\,\frac{\mathrm{d}s}{2\pi\mathrm{i}} \underset{n\to\infty}{=} \exp(-n(g^e(z)-\ell_e+Q_e))\Big((m_p^{a,2}(z))_{12}\Big(1+\frac{1}{n}\Big(\mathcal{R}_{\infty}^e(z)-\widetilde{\mathcal{R}}_{\infty}^e(z)\Big)_{11}$$
$$+\ \mathcal{O}\Big(\frac{1}{n^2}\Big)\Big)+(m_p^{a,2}(z))_{22}\Big(\frac{1}{n}\Big(\mathcal{R}_{\infty}^e(z)-\widetilde{\mathcal{R}}_{\infty}^e(z)\Big)_{12}+\mathcal{O}\Big(\frac{1}{n^2}\Big)\Big)\Big), \tag{2.101}$$

where

$$(m_p^{a,2}(z))_{11} := -\mathrm{i}\,\sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}n\xi_{a_j}^e(z)}\Big(\mathrm{i}\Big(-\omega\,\mathrm{Ai}(\omega p_a)(p_a)^{1/4}+\omega^2\,\mathrm{Ai}'(\omega p_a)(p_a)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{11}$$
$$-\Big(\omega\,\mathrm{Ai}(\omega p_a)(p_a)^{1/4}+\omega^2\,\mathrm{Ai}'(\omega p_a)(p_a)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{12}\mathrm{e}^{-\mathrm{i}n\mho_j^e}\Big), \tag{2.102}$$

$$(m_p^{a,2}(z))_{12} := \sqrt{\pi}\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}}\mathrm{e}^{-\frac{1}{2}n\xi_{a_j}^e(z)}\Big(\mathrm{i}\Big(\mathrm{Ai}(\omega^2p_a)(p_a)^{1/4}-\omega^2\,\mathrm{Ai}'(\omega^2p_a)(p_a)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{11}$$
$$\times\,\mathrm{e}^{\mathrm{i}n\mho_j^e}+\Big(\mathrm{Ai}(\omega^2p_a)(p_a)^{1/4}+\omega^2\,\mathrm{Ai}'(\omega^2p_a)(p_a)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{12}\Big), \tag{2.103}$$

$$(m_p^{a,2}(z))_{21} := -\mathrm{i}\,\sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}n\xi_{a_j}^e(z)}\Big(\mathrm{i}\Big(-\omega\,\mathrm{Ai}(\omega p_a)(p_a)^{1/4}+\omega^2\,\mathrm{Ai}'(\omega p_a)(p_a)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{21}$$
$$-\Big(\omega\,\mathrm{Ai}(\omega p_a)(p_a)^{1/4}+\omega^2\,\mathrm{Ai}'(\omega p_a)(p_a)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{22}\mathrm{e}^{-\mathrm{i}n\mho_j^e}\Big), \tag{2.104}$$

$$(m_p^{a,2}(z))_{22} := \sqrt{\pi}\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}}\mathrm{e}^{-\frac{1}{2}n\xi_{a_j}^e(z)}\Big(\mathrm{i}\Big(\mathrm{Ai}(\omega^2p_a)(p_a)^{1/4}-\omega^2\,\mathrm{Ai}'(\omega^2p_a)(p_a)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{21}$$
$$\times\,\mathrm{e}^{\mathrm{i}n\mho_j^e}+\Big(\mathrm{Ai}(\omega^2p_a)(p_a)^{1/4}+\omega^2\,\mathrm{Ai}'(\omega^2p_a)(p_a)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{22}\Big); \tag{2.105}$$

(11) *for* $z\in\Omega_{a_j}^{e,3}$ $(\subset\mathbb{C}_-\cap\mathbb{U}_{\delta_{a_j}}^e)$, $j=1,\dots,N+1$,

$$\boldsymbol{\pi}_{2n}(z) \underset{n\to\infty}{=} \exp(n(g^e(z)+Q_e))\bigg(\Big((m_p^{a,3}(z))_{11}-(m_p^{a,3}(z))_{12}\mathrm{e}^{4n\pi\mathrm{i}\int_z^{a_{N+1}^e}\psi_V^e(s)\,\mathrm{d}s}\Big)\Big(1+\frac{1}{n}(\mathcal{R}_{\infty}^e(z)$$
$$-\ \widetilde{\mathcal{R}}_{\infty}^e(z)\Big)_{11}+\mathcal{O}\Big(\frac{1}{n^2}\Big)\Big)+\Big((m_p^{a,3}(z))_{21}-(m_p^{a,3}(z))_{22}\mathrm{e}^{4n\pi\mathrm{i}\int_z^{a_{N+1}^e}\psi_V^e(s)\,\mathrm{d}s}\Big)\Big(\frac{1}{n}(\mathcal{R}_{\infty}^e(z)$$
$$-\ \widetilde{\mathcal{R}}_{\infty}^e(z)\Big)_{12}+\mathcal{O}\Big(\frac{1}{n^2}\Big)\Big)\bigg), \tag{2.106}$$

and

$$\int_{\mathbb{R}} \frac{\boldsymbol{\pi}_{2n}(s)\mathrm{e}^{-n\widetilde{V}(s)}}{s-z} \, \frac{\mathrm{d}s}{2\pi\mathrm{i}} \underset{n\to\infty}{=} \exp(-n(g^e(z)-\ell_e+Q_e))\Big((m_p^{a,3}(z))_{12}\Big(1+\frac{1}{n}\Big(\mathcal{R}^e_\infty(z)-\widetilde{\mathcal{R}}^e_\infty(z)\Big)_{11} + O\Big(\frac{1}{n^2}\Big)\Big)+(m_p^{a,3}(z))_{22}\Big(\frac{1}{n}\Big(\mathcal{R}^e_\infty(z)-\widetilde{\mathcal{R}}^e_\infty(z)\Big)_{12}+O\Big(\frac{1}{n^2}\Big)\Big)\Big), \tag{2.107}$$

where

$$(m_p^{a,3}(z))_{11} := -\mathrm{i}\sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}n\xi^e_{a_j}(z)}\Big(\mathrm{i}\Big(-\omega^2\,\mathrm{Ai}(\omega^2 p_a)(p_a)^{1/4}+\omega\,\mathrm{Ai}'(\omega^2 p_a)(p_a)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{11} - \Big(\omega^2\,\mathrm{Ai}(\omega^2 p_a)(p_a)^{1/4}+\omega\,\mathrm{Ai}'(\omega^2 p_a)(p_a)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{12}\mathrm{e}^{\mathrm{i}n\mho^e_j}\Big), \tag{2.108}$$

$$(m_p^{a,3}(z))_{12} := \sqrt{\pi}\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}}\mathrm{e}^{-\frac{1}{2}n\xi^e_{a_j}(z)}\Big(\mathrm{i}\Big(-\omega^2\,\mathrm{Ai}(\omega p_a)(p_a)^{1/4}+\mathrm{Ai}'(\omega p_a)(p_a)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{11} \times \mathrm{e}^{-\mathrm{i}n\mho^e_j}-\Big(\omega^2\,\mathrm{Ai}(\omega p_a)(p_a)^{1/4}+\mathrm{Ai}'(\omega p_a)(p_a)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{12}\Big), \tag{2.109}$$

$$(m_p^{a,3}(z))_{21} := -\mathrm{i}\sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}n\xi^e_{a_j}(z)}\Big(\mathrm{i}\Big(-\omega^2\,\mathrm{Ai}(\omega^2 p_a)(p_a)^{1/4}+\omega\,\mathrm{Ai}'(\omega^2 p_a)(p_a)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{21} - \Big(\omega^2\,\mathrm{Ai}(\omega^2 p_a)(p_a)^{1/4}+\omega\,\mathrm{Ai}'(\omega^2 p_a)(p_a)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{22}\mathrm{e}^{\mathrm{i}n\mho^e_j}\Big), \tag{2.110}$$

$$(m_p^{a,3}(z))_{22} := \sqrt{\pi}\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}}\mathrm{e}^{-\frac{1}{2}n\xi^e_{a_j}(z)}\Big(\mathrm{i}\Big(-\omega^2\,\mathrm{Ai}(\omega p_a)(p_a)^{1/4}+\mathrm{Ai}'(\omega p_a)(p_a)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{21} \times \mathrm{e}^{-\mathrm{i}n\mho^e_j}-\Big(\omega^2\,\mathrm{Ai}(\omega p_a)(p_a)^{1/4}+\mathrm{Ai}'(\omega p_a)(p_a)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{22}\Big); \tag{2.111}$$

and **(12)** *for* $z\in\Omega^{e,4}_{a_j}$ $(\subset\mathbb{C}_-\cap\mathbb{U}^e_{\delta_{a_j}})$, $j=1,\dots,N+1$,

$$\boldsymbol{\pi}_{2n}(z) \underset{n\to\infty}{=} \exp(n(g^e(z)+Q_e))\Big((m_p^{a,4}(z))_{11}\Big(1+\frac{1}{n}\Big(\mathcal{R}^e_\infty(z)-\widetilde{\mathcal{R}}^e_\infty(z)\Big)_{11}+O\Big(\frac{1}{n^2}\Big)\Big) + (m_p^{a,4}(z))_{21}\Big(\frac{1}{n}\Big(\mathcal{R}^e_\infty(z)-\widetilde{\mathcal{R}}^e_\infty(z)\Big)_{12}+O\Big(\frac{1}{n^2}\Big)\Big)\Big), \tag{2.112}$$

and

$$\int_{\mathbb{R}} \frac{\boldsymbol{\pi}_{2n}(s)\mathrm{e}^{-n\widetilde{V}(s)}}{s-z} \, \frac{\mathrm{d}s}{2\pi\mathrm{i}} \underset{n\to\infty}{=} \exp(-n(g^e(z)-\ell_e+Q_e))\Big((m_p^{a,4}(z))_{12}\Big(1+\frac{1}{n}\Big(\mathcal{R}^e_\infty(z)-\widetilde{\mathcal{R}}^e_\infty(z)\Big)_{11} + O\Big(\frac{1}{n^2}\Big)\Big)+(m_p^{a,4}(z))_{22}\Big(\frac{1}{n}\Big(\mathcal{R}^e_\infty(z)-\widetilde{\mathcal{R}}^e_\infty(z)\Big)_{12}+O\Big(\frac{1}{n^2}\Big)\Big)\Big), \tag{2.113}$$

where

$$(m_p^{a,4}(z))_{11} := -\mathrm{i}\sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}n\xi^e_{a_j}(z)}\Big(\mathrm{i}\Big(\mathrm{Ai}(p_a)(p_a)^{1/4}-\mathrm{Ai}'(p_a)(p_a)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{11} + \Big(\mathrm{Ai}(p_a)(p_a)^{1/4}+\mathrm{Ai}'(p_a)(p_a)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{12}\mathrm{e}^{\mathrm{i}n\mho^e_j}\Big), \tag{2.114}$$

$$(m_p^{a,4}(z))_{12} := \sqrt{\pi}\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}}\mathrm{e}^{-\frac{1}{2}n\xi^e_{a_j}(z)}\Big(\mathrm{i}\Big(-\omega^2\,\mathrm{Ai}(\omega p_a)(p_a)^{1/4}+\mathrm{Ai}'(\omega p_a)(p_a)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{11} \times \mathrm{e}^{-\mathrm{i}n\mho^e_j}-\Big(\omega^2\,\mathrm{Ai}(\omega p_a)(p_a)^{1/4}+\mathrm{Ai}'(\omega p_a)(p_a)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{12}\Big), \tag{2.115}$$

$$(m_p^{a,4}(z))_{21} := -\mathrm{i}\sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}n\xi^e_{a_j}(z)}\Big(\mathrm{i}\Big(\mathrm{Ai}(p_a)(p_a)^{1/4}-\mathrm{Ai}'(p_a)(p_a)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{21} + \Big(\mathrm{Ai}(p_a)(p_a)^{1/4}+\mathrm{Ai}'(p_a)(p_a)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{22}\mathrm{e}^{\mathrm{i}n\mho^e_j}\Big), \tag{2.116}$$

$$(m_p^{a,4}(z))_{22} := \sqrt{\pi}\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}}\mathrm{e}^{-\frac{1}{2}n\xi^e_{a_j}(z)}\Big(\mathrm{i}\Big(-\omega^2\,\mathrm{Ai}(\omega p_a)(p_b)^{1/4}+\mathrm{Ai}'(\omega p_a)(p_a)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{21} \times \mathrm{e}^{-\mathrm{i}n\mho^e_j}-\Big(\omega^2\,\mathrm{Ai}(\omega p_a)(p_a)^{1/4}+\mathrm{Ai}'(\omega p_a)(p_a)^{-1/4}\Big)(\overset{e}{m}{}^{\infty}(z))_{22}\Big). \tag{2.117}$$

Remark 2.3.2. Using limiting values, if necessary, all of the above (asymptotic) formulae for $\boldsymbol{\pi}_{2n}(z)$ and $\int_{\mathbb{R}}\boldsymbol{\pi}_{2n}(s)\mathrm{e}^{-n\widetilde{V}(s)}(s-z)^{-1}\frac{\mathrm{d}s}{2\pi\mathrm{i}}$ have a natural interpretation on the real and imaginary axes. ■

Theorem 2.3.2. *Let all the conditions stated in Theorem* 2.3.1 *be valid, and let* $\overset{e}{\mathrm{Y}}\colon\mathbb{C}\setminus\mathbb{R}\to\mathrm{SL}_2(\mathbb{C})$ *be the unique solution of* **RHP1**. *Let* $H^{(m)}_k$, $(m,k)\in\mathbb{Z}\times\mathbb{N}$, *be the Hankel determinants associated with the bi-infinite,*

real-valued, strong moment sequence $\left\{c_k = \int_{\mathbb{R}} s^k \mathrm{e}^{-n\widetilde{V}(s)}\,\mathrm{d}s,\ n\in\mathbb{N}\right\}_{k\in\mathbb{Z}}$ *defined in Equations* (1.1)*, and let* $\boldsymbol{\pi}_{2n}(z)$ *be the even degree monic orthogonal L-polynomial defined in Lemma* 2.2.1*, that is,* $\boldsymbol{\pi}_{2n}(z) := (\overset{e}{\mathrm{Y}}(z))_{11}$*, with* $n\to\infty$ *asymptotics (in the entire complex plane) given by Theorem* 2.3.1*. Then,*

$$(\xi_n^{(2n)})^2 = \frac{1}{\|\boldsymbol{\pi}_{2n}(\cdot)\|_{\mathcal{L}}^2} = \frac{H_{2n}^{(-2n)}}{H_{2n+1}^{(-2n)}} \underset{n\to\infty}{=} \frac{\mathrm{e}^{-n\ell_e}}{\pi}\,\Xi^{\flat}\left(1+\frac{1}{n}\Xi^{\flat}(\mathfrak{Q}^{\flat})_{12}+O\left(\frac{c^{\flat}(n)}{n^2}\right)\right), \tag{2.118}$$

where

$$\Xi^{\flat} := 2\left(\sum_{k=1}^{N+1}\left(a_k^e - b_{k-1}^e\right)\right)^{-1} \frac{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\infty)-\frac{n}{2\pi}\boldsymbol{\Omega}^e+\boldsymbol{d}_e)\boldsymbol{\theta}^e(-\boldsymbol{u}_+^e(\infty)+\boldsymbol{d}_e)}{\boldsymbol{\theta}^e(-\boldsymbol{u}_+^e(\infty)-\frac{n}{2\pi}\boldsymbol{\Omega}^e+\boldsymbol{d}_e)\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\infty)+\boldsymbol{d}_e)}, \tag{2.119}$$

$$\mathfrak{Q}^{\flat} := 2\mathrm{i}\sum_{j=1}^{N+1}\left(\frac{(\mathcal{B}^e(a_j^e)\widehat{\alpha}_0^e(a_j^e)-\mathcal{A}^e(a_j^e)\widehat{\alpha}_1^e(a_j^e))}{(\widehat{\alpha}_0^e(a_j^e))^2}+\frac{(\mathcal{B}^e(b_{j-1}^e)\widehat{\alpha}_0^e(b_{j-1}^e)-\mathcal{A}^e(b_{j-1}^e)\widehat{\alpha}_1^e(b_{j-1}^e))}{(\widehat{\alpha}_0^e(b_{j-1}^e))^2}\right), \tag{2.120}$$

$(\mathfrak{Q}^{\flat})_{12}$ *denotes the* $(1\,2)$*-element of* $\mathfrak{Q}^{\flat}$*,* $c^{\flat}(n) =_{n\to\infty} O(1)$*, and all the relevant parameters are defined in Theorem* 2.3.1*: asymptotics for* $\xi_n^{(2n)}$ *are obtained by taking the positive square root of both sides of Equation* (2.118)*. Furthermore, the* $n\to\infty$ *asymptotic expansion (in the entire complex plane) for the even degree orthonormal L-polynomial,*

$$\phi_{2n}(z) = \xi_n^{(2n)}\boldsymbol{\pi}_{2n}(z), \tag{2.121}$$

to $O(n^{-2})$*, is given by the (scalar) multiplication of the* $n\to\infty$ *asymptotics of* $\boldsymbol{\pi}_{2n}(z)$ *and* $\xi_n^{(2n)}$ *stated, respectively, in Theorem* 2.3.1 *and Equations* (2.118)–(2.120)*.*

Remark 2.3.3. Since, from general theory (cf. Section 1), and, by construction (cf. Equations (1.2) and (1.8)), $\xi_n^{(2n)} > 0$, it follows, incidentally, from Theorem 2.3.2, Equations (2.118)–(2.120), that: (i) $\Xi^{\flat} > 0$; and (ii) $\mathrm{Im}((\mathfrak{Q}^{\flat})_{12}) = 0$. ■

3 The Equilibrium Measure, the Variational Problem, and the Transformed RHP

In this section, the detailed analysis of the 'even degree' variational problem, and the associated 'even' equilibrium measure, is undertaken (see Lemmas 3.1–3.3 and Lemma 3.5), including the discussion of the corresponding g-function, denoted, herein, as g^e, and **RHP1**, that is, $(\overset{e}{\mathrm{Y}}(z), \mathrm{I}+\exp(-n\widetilde{V}(z))\sigma_+, \mathbb{R})$, is reformulated as an equivalent[6], auxiliary RHP (see Lemma 3.4). The proofs of Lemmas 3.1–3.3 are modelled on the calculations of Saff-Totik ([55], Chapter 1), Deift ([90], Chapter 6), and Johansson [91].

One begins by establishing the existence of the 'even' equilibrium measure, μ_V^e $(\in\mathcal{M}_1(\mathbb{R}))$.

Lemma 3.1. *Let the external field* $\widetilde{V}\colon \mathbb{R}\setminus\{0\}\to\mathbb{R}$ *satisfy conditions* (2.3)–(2.5)*, and set* $w^e(z) := \mathrm{e}^{-\widetilde{V}(z)}$*. For* $\mu^e\in\mathcal{M}_1(\mathbb{R})$*, define the weighted energy functional* $\mathrm{I}_V^e[\mu^e]\colon \mathcal{M}_1(\mathbb{R})\to\mathbb{R}$*,*

$$\mathrm{I}_V^e[\mu^e] := \iint_{\mathbb{R}^2}\ln\left(|s-t|^2|st|^{-1}w^e(s)w^e(t)\right)^{-1}\mathrm{d}\mu^e(s)\,\mathrm{d}\mu^e(t),$$

and consider the minimisation problem

$$E_V^e = \inf\left\{\mathrm{I}_V^e[\mu^e];\ \mu^e\in\mathcal{M}_1(\mathbb{R})\right\}.$$

Then: (1) E_V^e *is finite;* (2) $\exists\ \mu_V^e\in\mathcal{M}_1(\mathbb{R})$ *such that* $\mathrm{I}_V^e[\mu_V^e] = E_V^e$ *(the infimum is attained), and* μ_V^e *has finite weighted logarithmic energy* $(-\infty < \mathrm{I}_V^e[\mu_V^e] < +\infty)$*; and* (3) $J_e := \mathrm{supp}(\mu_V^e)$ *is compact,* $J_e\subset\{z;\ w^e(z)>0\}$*, and* J_e *has positive logarithmic capacity, that is,* $\mathrm{cap}(J_e) := \exp\left(-\inf\{\mathrm{I}_V^e[\mu^e];\ \mu^e\in\mathcal{M}_1(J_e)\}\right) > 0$*.*

[6]If there are two RHPs, $(\mathcal{Y}_1(z), v_1(z), \Gamma_1)$ and $(\mathcal{Y}_2(z), v_2(z), \Gamma_2)$, say, with $\Gamma_2\subset\Gamma_1$ and $v_1(z)\restriction_{\Gamma_1\setminus\Gamma_2} =_{n\to\infty} \mathrm{I}+o(1)$, then, within the BC framework [84], and modulo $o(1)$ estimates, their solutions, $\mathcal{Y}_1$ and $\mathcal{Y}_2$, respectively, are (asymptotically) equal.

Proof. Let $\mu^e \in \mathcal{M}_1(\mathbb{R})$, and set $w^e(z) := \exp(-\widetilde{V}(z))$, where $\widetilde{V} \colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ satisfies conditions (2.3)–(2.5). From the definition of $\mathrm{I}_V^e[\mu^e]$ given in the Lemma, one shows that

$$\begin{aligned}\mathrm{I}_V^e[\mu^e] &= \iint_{\mathbb{R}^2} \left(\ln(|s-t|^{-1}) + \ln(|s^{-1}-t^{-1}|^{-1})\right) \mathrm{d}\mu^e(s)\,\mathrm{d}\mu^e(t) + 2\int_{\mathbb{R}} \widetilde{V}(s)\,\mathrm{d}\mu^e(s)\\ &=: \iint_{\mathbb{R}^2} K_V^e(s,t)\,\mathrm{d}\mu^e(s)\,\mathrm{d}\mu^e(t),\end{aligned}$$

where (the symmetric kernel)

$$K_V^e(s,t) = K_V^e(t,s) := \ln(|s-t|^{-1}) + \ln(|s^{-1}-t^{-1}|^{-1}) + \widetilde{V}(s) + \widetilde{V}(t)$$

(of course, the definition of $\mathrm{I}_V^e[\mu^e]$ only makes sense provided both integrals exist and are finite). Recall the following inequalities (see, for example, Chapter 6 of [90]): $|s-t| \leqslant (1+s^2)^{1/2}(1+t^2)^{1/2}$ and $|s^{-1}-t^{-1}| \leqslant (1+s^{-2})^{1/2}(1+t^{-2})^{1/2}$, $s,t \in \mathbb{R}$, whence $\ln(|s-t|^{-1}) \geqslant -\frac{1}{2}\ln(1+s^2) - \frac{1}{2}\ln(1+t^2)$ and $\ln(|s^{-1}-t^{-1}|^{-1}) \geqslant -\frac{1}{2}\ln(1+s^{-2}) - \frac{1}{2}\ln(1+t^{-2})$; thus,

$$K_V^e(s,t) \geqslant \frac{1}{2}\left(2\widetilde{V}(s) - \ln(s^2+1) - \ln(s^{-2}+1)\right) + \frac{1}{2}\left(2\widetilde{V}(t) - \ln(t^2+1) - \ln(t^{-2}+1)\right).$$

Recalling conditions (2.3)–(2.5) for the external field $\widetilde{V} \colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$, in particular, $\exists\ \delta_1 > 0$ (resp., $\exists\ \delta_2 > 0$) such that $\widetilde{V}(x) \geqslant (1+\delta_1)\ln(x^2+1)$ (resp., $\widetilde{V}(x) \geqslant (1+\delta_2)\ln(x^{-2}+1)$) for sufficiently large $|x|$ (resp., small $|x|$), it follows that $2\widetilde{V}(x) - \ln(x^2+1) - \ln(x^{-2}+1) \geqslant C_V^e > -\infty$, whence $K_V^e(s,t) \geqslant C_V^e$ $(>-\infty)$, which shows that $K_V^e(s,t)$ is bounded from below (on $\mathbb{R}^2$); hence,

$$\mathrm{I}_V^e[\mu^e] \geqslant \iint_{\mathbb{R}^2} C_V^e\,\mathrm{d}\mu^e(s)\,\mathrm{d}\mu^e(t) = C_V^e \underbrace{\int_{\mathbb{R}} \mathrm{d}\mu^e(s)}_{=1} \underbrace{\int_{\mathbb{R}} \mathrm{d}\mu^e(t)}_{=1} \geqslant C_V^e \quad (>-\infty).$$

It follows from the above inequality and the definition of E_V^e stated in the Lemma that, $\forall\ \mu^e \in \mathcal{M}_1(\mathbb{R})$, $E_V^e \geqslant C_V^e > -\infty$, which shows that E_V^e is bounded from below. Let ε be an arbitrarily fixed, sufficiently small positive real number, and set $\Sigma_{e,\varepsilon} := \{z;\ w^e(z) \geqslant \varepsilon\}$; then, $\Sigma_{e,\varepsilon}$ is compact, and $\Sigma_{e,0} := \cup_{l=1}^{\infty} \Sigma_{e,1/l} = \cup_{l=1}^{\infty}\{z;\ w^e(z) > l^{-1}\} = \{z;\ w^e(z) > 0\}$. Since, for $\widetilde{V} \colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ satisfying conditions (2.3)–(2.5), w^e is an *admissible weight* [55], in which case $\Sigma_{e,0}$ has positive logarithmic capacity, that is, $\mathrm{cap}(\Sigma_{e,0}) = \exp(-\inf\{\mathrm{I}_V^e[\mu^e];\ \mu^e \in \mathcal{M}_1(\Sigma_{e,0})\}) > 0$, it follows that $\exists\ l^* \in \mathbb{N}$ such that $\mathrm{cap}(\Sigma_{e,1/l^*}) = \exp(-\inf\{\mathrm{I}_V^e[\mu^e];\ \mu^e \in \mathcal{M}_1(\Sigma_{e,1/l^*})\}) > 0$, which, in turn, means that there exists a probability measure, $\mu_{l^*}^e$, say, with $\mathrm{supp}(\mu_{l^*}^e) \subseteq \Sigma_{e,1/l^*}$, such that $\iint_{\Sigma_{e,1/l^*}^2} \ln(|s-t|^{-2}|st|)\,\mathrm{d}\mu_{l^*}^e(s)\,\mathrm{d}\mu_{l^*}^e(t) < +\infty$, where $\Sigma_{e,1/l^*}^2 = \Sigma_{e,1/l^*} \times \Sigma_{e,1/l^*}$ $(\subseteq \mathbb{R}^2)$. For $z \in \mathrm{supp}(\mu_{l^*}^e) \subseteq \Sigma_{e,1/l^*}$, it follows that $w^e(z) \geqslant 1/l^*$, whence $\iint_{\Sigma_{e,1/l^*}^2} \ln(w^e(s)w^e(t))^{-1}\,\mathrm{d}\mu_{l^*}^e(s)\,\mathrm{d}\mu_{l^*}^e(t) \leqslant 2\ln(l^*) < +\infty$ $\Rightarrow$

$$\mathrm{I}_V^e[\mu_{l^*}^e] = \iint_{\Sigma_{e,1/l^*}^2} \ln\left(|s-t|^2|st|^{-1}w^e(s)w^e(t)\right)^{-1} \mathrm{d}\mu_{l^*}^e(s)\,\mathrm{d}\mu_{l^*}^e(t) < +\infty;$$

thus, it follows that $E_V^e := \inf\{\mathrm{I}_V^e[\mu^e];\ \mu^e \in \mathcal{M}_1(\mathbb{R})\}$ is finite (see, also, below).

Choose a sequence of probability measures $\{\mu_n^e\}_{n=1}^{\infty}$ in $\mathcal{M}_1(\mathbb{R})$ such that $\mathrm{I}_V^e[\mu_n^e] \leqslant E_V^e + \frac{1}{n}$. From the analysis above, it follows that

$$\begin{aligned}\mathrm{I}_V^e[\mu_n^e] &= \iint_{\mathbb{R}^2} K_V^e(s,t)\,\mathrm{d}\mu_n^e(s)\,\mathrm{d}\mu_n^e(t) \geqslant \iint_{\mathbb{R}^2} \Big(\tfrac{1}{2}(2\widetilde{V}(s) - \ln(s^2+1) - \ln(s^{-2}+1))\\ &\quad + \tfrac{1}{2}(2\widetilde{V}(t) - \ln(t^2+1) - \ln(t^{-2}+1))\Big)\mathrm{d}\mu_n^e(s)\,\mathrm{d}\mu_n^e(t).\end{aligned}$$

Set

$$\widehat{\psi}_V^e(z) := 2\widetilde{V}(z) - \ln(z^2+1) - \ln(z^{-2}+1).$$

Then $\mathrm{I}_V^e[\mu_n^e] \geqslant \int_{\mathbb{R}} \widehat{\psi}_V^e(s)\,\mathrm{d}\mu_n^e(s) \Rightarrow E_V^e + \frac{1}{n} \geqslant \int_{\mathbb{R}} \widehat{\psi}_V^e(s)\,\mathrm{d}\mu_n^e(s)$. Recalling that $\exists\ \delta_1 > 0$ (resp., $\exists\ \delta_2 > 0$) such that $\widetilde{V}(x) \geqslant (1+\delta_1)\ln(x^2+1)$ (resp., $\widetilde{V}(x) \geqslant (1+\delta_2)\ln(x^{-2}+1)$) for sufficiently large $|x|$ (resp., small $|x|$), it

follows that, for any $b_e>0$, $\exists\, M_e>1$ such that $\widehat{\psi}_V^e(z)>b_e$ $\forall\, z\in\{|z|\geqslant M_e\}\cup\{|z|\leqslant M_e^{-1}\}=:\mathfrak{D}_e$, which implies that

$$E_V^e+\tfrac{1}{n}\geqslant\int_{\mathbb{R}}\widehat{\psi}_V^e(s)\,\mathrm{d}\mu_n^e(s)=\int_{\mathfrak{D}_e}\underbrace{\widehat{\psi}_V^e(s)}_{>b_e}\,\mathrm{d}\mu_n^e(s)+\int_{\mathbb{R}\setminus\mathfrak{D}_e}\underbrace{\widehat{\psi}_V^e(s)}_{\geqslant-|C_V^e|}\,\mathrm{d}\mu_n^e(s)$$

$$\geqslant b_e\int_{\mathfrak{D}_e}\mathrm{d}\mu_n^e(s)-|C_V^e|\underbrace{\int_{\mathbb{R}\setminus\mathfrak{D}_e}\mathrm{d}\mu_n^e(s)}_{\in[0,1]}\geqslant b_e\int_{\mathfrak{D}_e}\mathrm{d}\mu_n^e(s)-|C_V^e|;$$

thus,

$$\int_{\mathfrak{D}_e}\mathrm{d}\mu_n^e(s)\leqslant b_e^{-1}\Big(E_V^e+|C_V^e|+\frac{1}{n}\Big),$$

whence

$$\limsup_{n\to\infty}\int_{\mathfrak{D}_e}\mathrm{d}\mu_n^e(s)\leqslant\limsup_{n\to\infty}\Big(b_e^{-1}\Big(E_V^e+|C_V^e|+\frac{1}{n}\Big)\Big).$$

By the Archimedean property, it follows that, $\forall\,\epsilon_o>0$, $\exists\,N\in\mathbb{N}$ such that, $\forall\,n>N\Rightarrow n^{-1}<\epsilon_o$; thus, choosing $b_e=\epsilon^{-1}(E_V^e+|C_V^e|+\epsilon_o)$, where ϵ is some arbitrarily fixed, sufficiently small positive real number, it follows that $\limsup_{n\to\infty}\int_{\mathfrak{D}_e}\mathrm{d}\mu_n^e(s)\leqslant\epsilon\Rightarrow$ the sequence of probability measures $\{\mu_n^e\}_{n=1}^{\infty}$ in $\mathcal{M}_1(\mathbb{R})$ is *tight* [91] (that is, given $\epsilon>0$, $\exists\,M>1$ such that $\limsup_{n\to\infty}\mu_n^e(\{|s|\geqslant M\}\cup\{|s|\leqslant M^{-1}\}):=\limsup_{n\to\infty}\int_{\{|s|\geqslant M\}\cup\{|s|\leqslant M^{-1}\}}\mathrm{d}\mu_n^e(s)\leqslant\epsilon$). Since the sequence of probabilty measures $\{\mu_n^e\}_{n=1}^{\infty}$ in $\mathcal{M}_1(\mathbb{R})$ is tight, by a Helly Selection Theorem, there exists a (weak* convergent) subsequence of probability measures $\{\mu_{n_k}^e\}_{k=1}^{\infty}$ in $\mathcal{M}_1(\mathbb{R})$ converging (weakly) to a probability measure $\mu^e\in\mathcal{M}_1(\mathbb{R})$, symbolically $\mu_{n_k}^e\overset{*}{\to}\mu^e$ as $k\to\infty$[7]. One now shows that, if $\mu_n^e\overset{*}{\to}\mu^e$, $\mu_n^e,\mu^e\in\mathcal{M}_1(\mathbb{R})$, then $\liminf_{n\to\infty}\mathrm{I}_V^e[\mu_n^e]\geqslant\mathrm{I}_V^e[\mu^e]$. Since w^e is continuous, thus upper semi-continuous [55], there exists a sequence $\{w_m^e\}_{m=1}^{\infty}$ (resp., $\{\widetilde{V}_m\}_{m=1}^{\infty}$) of continuous functions on $\mathbb{R}$ such that $w_{m+1}^e\leqslant w_m^e$ (resp., $\widetilde{V}_{m+1}\geqslant\widetilde{V}_m$)[8], $m\in\mathbb{N}$, and $w_m^e(z)\searrow w^e(z)$ (resp., $\widetilde{V}_m(z)\nearrow\widetilde{V}(z)$) as $m\to\infty$ for every $z\in\mathbb{R}$; in particular,

$$\mathrm{I}_V^e[\mu_n^e]=\iint_{\mathbb{R}^2}K_V^e(s,t)\,\mathrm{d}\mu_n^e(s)\,\mathrm{d}\mu_n^e(t)\geqslant\iint_{\mathbb{R}^2}K_{V_m}^e(s,t)\,\mathrm{d}\mu_n^e(s)\,\mathrm{d}\mu_n^e(t).$$

For arbitrary $q\in\mathbb{R}$, $\mathrm{I}_V^e[\mu_n^e]\geqslant\iint_{\mathbb{R}^2}p^e(s,t)\,\mathrm{d}\mu_n^e(s)\,\mathrm{d}\mu_n^e(t)$, where $p^e(s,t)=p^e(t,s):=\min\big\{q,K_{V_m}^e(s,t)\big\}$ (bounded and continuous on $\mathbb{R}^2$). Recall that $\{\mu_n^e\}_{n=1}^{\infty}$ is tight in $\mathcal{M}_1(\mathbb{R})$. For $M_e>1$, let $h_M^e(x)\in\mathrm{C}_{\mathbf{b}}^{\mathbf{0}}(\mathbb{R})$ be such that:

(i) $h_M^e(x)=1$, $x\in[-M_e,-M_e^{-1}]\cup[M_e^{-1},M_e]=:\mathfrak{D}_{M_e}$;

(ii) $h_M^e(x)=0$, $x\in\mathbb{R}\setminus\mathfrak{D}_{M_e+1}$; and

(iii) $0\leqslant h_M^e(x)\leqslant1$, $x\in\mathbb{R}$.

Note the decomposition $\iint_{\mathbb{R}^2}p^e(t,s)\,\mathrm{d}\mu_n^e(t)\,\mathrm{d}\mu_n^e(s)=I_a+I_b+I_c$, where

$$I_a:=\iint_{\mathbb{R}^2}p^e(t,s)(1-h_M^e(s))\,\mathrm{d}\mu_n^e(t)\,\mathrm{d}\mu_n^e(s),$$

$$I_b:=\iint_{\mathbb{R}^2}p^e(t,s)h_M^e(s)(1-h_M^e(t))\,\mathrm{d}\mu_n^e(t)\,\mathrm{d}\mu_n^e(s),$$

$$I_c:=\iint_{\mathbb{R}^2}p^e(t,s)h_M^e(t)h_M^e(s)\,\mathrm{d}\mu_n^e(t)\,\mathrm{d}\mu_n^e(s).$$

One shows that

$$|I_a|\leqslant\iint_{\mathbb{R}^2}|p^e(t,s)|(1-h_M^e(s))\,\mathrm{d}\mu_n^e(t)\,\mathrm{d}\mu_n^e(s)$$

[7] A sequence of probability measures $\{\mu_n\}_{n=1}^{\infty}$ in $\mathcal{M}_1(D)$ is said to *converge weakly* as $n\to\infty$ to $\mu\in\mathcal{M}_1(D)$, symbolically $\mu_n\overset{*}{\to}\mu$, if $\mu_n(f):=\int_D f(s)\,\mathrm{d}\mu_n(s)\to\int_D f(s)\,\mathrm{d}\mu(s)=:\mu(f)$ as $n\to\infty$ $\forall\ f\in\mathrm{C}_{\mathbf{b}}^{\mathbf{0}}(D)$, where $\mathrm{C}_{\mathbf{b}}^{\mathbf{0}}(D)$ denotes the set of all bounded, continuous functions on D with compact support.

[8] Adding a suitable constant, if necessary, which does not change μ_m^e, or the regularity of $\widetilde{V}\colon\mathbb{R}\setminus\{0\}\to\mathbb{R}$, one may assume that $\widetilde{V}\geqslant0$ and $\widetilde{V}_m\geqslant0$, $m\in\mathbb{N}$.

$$\leqslant \sup_{(t,s)\in\mathbb{R}^2}|p^e(t,s)| \underbrace{\int_{\mathbb{R}} \mathrm{d}\mu_n^e(t)}_{=1}\left(\int_{\mathfrak{D}_{M_e}} \underbrace{(1-h_M^e(s))}_{=0}\,\mathrm{d}\mu_n^e(s)+\int_{\mathbb{R}\setminus\mathfrak{D}_{M_e+1}}(1-\underbrace{h_M^e(s))}_{=0}\,\mathrm{d}\mu_n^e(s)\right),$$

whence

$$\limsup_{n\to\infty}|I_a|\leqslant \sup_{(t,s)\in\mathbb{R}^2}|p^e(t,s)|\underbrace{\limsup_{n\to\infty}\int_{\mathbb{R}\setminus\mathfrak{D}_{M_e+1}}\mathrm{d}\mu_n^e(s)}_{\leqslant\epsilon}\leqslant\epsilon\sup_{(t,s)\in\mathbb{R}^2}|p^e(t,s)|;$$

similarly,

$$\limsup_{n\to\infty}|I_b|\leqslant\epsilon\sup_{(t,s)\in\mathbb{R}^2}|p^e(t,s)|.$$

Since $p^e(t,s)$ is continuous and bounded on $\mathbb{R}^2$, there exists, by a generalisation of the Stone-Weierstrass Theorem (for the single-variable case), a polynomial in two variables, $p(t,s)$, say, with $p(t,s)=\sum_{i\geqslant i_o}\sum_{j\geqslant j_o}\gamma_{ij}t^is^j$, such that $|p^e(t,s)-p(t,s)|\leqslant\epsilon$; thus,

$$|h_M^e(t)h_M^e(s)p^e(t,s)-h_M^e(t)h_M^e(s)p(t,s)|\leqslant\epsilon,\quad t,s\in\mathbb{R}.$$

Rewrite I_c as

$$\begin{aligned}I_c &= \iint_{\mathbb{R}^2} h_M^e(s)h_M^e(t)p(t,s)\,\mathrm{d}\mu_n^e(t)\,\mathrm{d}\mu_n^e(s)+\iint_{\mathbb{R}^2} h_M^e(s)h_M^e(t)(p^e(t,s)-p(t,s))\,\mathrm{d}\mu_n^e(t)\,\mathrm{d}\mu_n^e(s)\\ &=: I_c^\alpha+I_c^\beta.\end{aligned}$$

One now shows that

$$\begin{aligned}|I_c^\beta| &\leqslant \iint_{\mathbb{R}^2} h_M^e(s)h_M^e(t)\underbrace{|p^e(t,s)-p(t,s)|}_{\leqslant\epsilon}\,\mathrm{d}\mu_n^e(t)\,\mathrm{d}\mu_n^e(s)\leqslant\epsilon\int_{\mathbb{R}}h_M^e(s)\,\mathrm{d}\mu_n^e(s)\int_{\mathbb{R}}h_M^e(t)\,\mathrm{d}\mu_n^e(t)\\ &\leqslant\epsilon\left(\int_{\mathfrak{D}_{M_e}}\underbrace{h_M^e(s)}_{=1}\,\mathrm{d}\mu_n^e(s)+\int_{\mathbb{R}\setminus\mathfrak{D}_{M_e+1}}\underbrace{h_M^e(s)}_{=0}\,\mathrm{d}\mu_n^e(s)\right)^2\leqslant\epsilon\left(\int_{\mathfrak{D}_{M_e}}\mathrm{d}\mu_n^e(s)\right)^2\\ &\leqslant\epsilon\left(\int_{\mathbb{R}}\mathrm{d}\mu_n^e(s)\right)^2\leqslant\epsilon,\end{aligned}$$

and

$$\begin{aligned}I_c^\alpha &= \iint_{\mathbb{R}^2} h_M^e(s)h_M^e(t)\sum_{i\geqslant i_o}\sum_{j\geqslant j_o}\gamma_{ij}t^is^j\,\mathrm{d}\mu_n^e(t)\,\mathrm{d}\mu_n^e(s)\\ &= \sum_{i\geqslant i_o}\sum_{j\geqslant j_o}\gamma_{ij}\left(\int_{\mathbb{R}}h_M^e(t)t^i\,\mathrm{d}\mu_n^e(t)\right)\left(\int_{\mathbb{R}}h_M^e(s)s^j\,\mathrm{d}\mu_n^e(s)\right)\\ &\to \sum_{i\geqslant i_o}\sum_{j\geqslant j_o}\gamma_{ij}\left(\int_{\mathbb{R}}h_M^e(t)t^i\,\mathrm{d}\mu^e(t)\right)\left(\int_{\mathbb{R}}h_M^e(s)s^j\,\mathrm{d}\mu^e(s)\right)\qquad(\text{since } \mu_n^e\overset{*}{\to}\mu^e \text{ as } n\to\infty)\\ &= \iint_{\mathbb{R}^2}\left(\sum_{i\geqslant i_o}\sum_{j\geqslant j_o}\gamma_{ij}t^is^j\right)h_M^e(t)h_M^e(s)\,\mathrm{d}\mu^e(t)\,\mathrm{d}\mu^e(s),\end{aligned}$$

whence, recalling that $p(t,s)=\sum_{i\geqslant i_o}\sum_{j\geqslant j_o}\gamma_{ij}t^is^j$, it follows that

$$I_c^\alpha=\iint_{\mathbb{R}^2}p(t,s)h_M^e(t)h_M^e(s)\,\mathrm{d}\mu^e(t)\,\mathrm{d}\mu^e(s).$$

Furthermore,

$$I_c^\alpha \leqslant \iint_{\mathbb{R}^2} p^e(t,s) h_M^e(t) h_M^e(s)\,\mathrm{d}\mu^e(t)\,\mathrm{d}\mu^e(s) + \epsilon \underbrace{\int_{\mathbb{R}} h_M^e(t)\,\mathrm{d}\mu^e(t)}_{\leqslant 1} \underbrace{\int_{\mathbb{R}} h_M^e(s)\,\mathrm{d}\mu^e(s)}_{\leqslant 1} \Rightarrow$$

$$\begin{aligned}
I_c^\alpha &\leqslant \iint_{\mathbb{R}^2} p^e(t,s)|1+(h_M^e(t)-1)||1+(h_M^e(s)-1)|\,\mathrm{d}\mu^e(t)\,\mathrm{d}\mu^e(s)+\epsilon \\
&\leqslant \iint_{\mathbb{R}^2} p^e(t,s)\,\mathrm{d}\mu^e(t)\,\mathrm{d}\mu^e(s) + \iint_{\mathbb{R}^2} p^e(t,s)|h_M^e(s)-1|\,\mathrm{d}\mu^e(t)\,\mathrm{d}\mu^e(s)+\epsilon \\
&+ \iint_{\mathbb{R}^2} p^e(t,s)|h_M^e(t)-1|\,\mathrm{d}\mu^e(t)\,\mathrm{d}\mu^e(s) + \iint_{\mathbb{R}^2} p^e(t,s)|h_M^e(t)-1||h_M^e(s)-1|\,\mathrm{d}\mu^e(t)\,\mathrm{d}\mu^e(s) \\
&\leqslant \iint_{\mathbb{R}^2} p^e(t,s)\,\mathrm{d}\mu^e(t)\,\mathrm{d}\mu^e(s) + 2\sup_{(t,s)\in\mathbb{R}^2}|p^e(t,s)| \underbrace{\int_{(\mathbb{R}\setminus\mathfrak{D}_{M_e})\cup\mathfrak{D}_{M_e}} |h_M^e(s)-1|\,\mathrm{d}\mu^e(s)}_{\leqslant\epsilon} \underbrace{\int_{\mathbb{R}}\mathrm{d}\mu^e(t)}_{=1} \\
&+ \sup_{(t,s)\in\mathbb{R}^2}|p^e(t,s)| \left(\underbrace{\int_{(\mathbb{R}\setminus\mathfrak{D}_{M_e})\cup\mathfrak{D}_{M_e}} |h_M^e(t)-1|\,\mathrm{d}\mu^e(t)}_{\leqslant\epsilon}\right)^2 + \epsilon \\
&\leqslant \iint_{\mathbb{R}^2} p^e(t,s)\,\mathrm{d}\mu^e(t)\,\mathrm{d}\mu^e(s) + \epsilon\left(1+2\sup_{(t,s)\in\mathbb{R}^2}|p^e(t,s)|\right) + O(\epsilon^2),
\end{aligned}$$

whereupon, neglecting the $O(\epsilon^2)$ term, and setting $\varkappa^\flat := 1+2\sup_{(t,s)\in\mathbb{R}^2}|p^e(t,s)|$, one obtains

$$I_c^\alpha \leqslant \iint_{\mathbb{R}^2} p^e(t,s)\,\mathrm{d}\mu^e(t)\,\mathrm{d}\mu^e(s) + \varkappa^\flat\epsilon.$$

Hence, assembling the above-derived bounds for I_a, I_b, I_c^β, and I_c^α, one arrives at, upon setting $\epsilon^\flat := 2\varkappa^\flat\epsilon$,

$$\iint_{\mathbb{R}^2} p^e(t,s)\,\mathrm{d}\mu_n^e(t)\,\mathrm{d}\mu_n^e(s) - \iint_{\mathbb{R}^2} p^e(t,s)\,\mathrm{d}\mu^e(t)\,\mathrm{d}\mu^e(s) \leqslant \epsilon^\flat;$$

thus,

$$\iint_{\mathbb{R}^2} p^e(t,s)\,\mathrm{d}\mu_n^e(t)\,\mathrm{d}\mu_n^e(s) \to \iint_{\mathbb{R}^2} p^e(t,s)\,\mathrm{d}\mu^e(t)\,\mathrm{d}\mu^e(s) \quad \text{as} \ \ n\to\infty.$$

Recalling that $p^e(t,s) := \min\left\{q, K_{V_m}^e(t,s)\right\}$, $(q,m)\in\mathbb{R}\times\mathbb{N}$, it follows from the above analysis that

$$\liminf_{n\to\infty} \mathrm{I}_V^e[\mu_n^e] \geqslant \iint_{\mathbb{R}^2} \min\left\{q, K_{V_m}^e(t,s)\right\}\,\mathrm{d}\mu^e(t)\,\mathrm{d}\mu^e(s):$$

letting $q\uparrow\infty$ and $m\to\infty$, and using the Monotone Convergence Theorem, one arrives at, upon noting that $\min\left\{q, K_{V_m}^e(t,s)\right\}\to K_V^e(t,s)$,

$$\liminf_{n\to\infty} \mathrm{I}_V^e[\mu_n^e] \geqslant \iint_{\mathbb{R}^2} K_V^e(t,s)\,\mathrm{d}\mu^e(t)\,\mathrm{d}\mu^e(s) = \mathrm{I}_V^e[\mu^e], \qquad \mu_n^e,\mu^e\in\mathcal{M}_1(\mathbb{R}).$$

Since, from the analysis above, it was shown that there exists a weakly (weak*) convergent subsequence (of probability measures) $\{\mu_{n_k}^e\}_{k=1}^\infty$ $(\subset\mathcal{M}_1(\mathbb{R}))$ of $\{\mu_n^e\}_{n=1}^\infty$ $(\subset\mathcal{M}_1(\mathbb{R}))$ with a weak limit $\mu^e\in\mathcal{M}_1(\mathbb{R})$, namely, $\mu_{n_k}^e\to\mu^e$ as $k\to\infty$, upon recalling that $\mathrm{I}_V^e[\mu_n^e]\leqslant E_V^e+\frac{1}{n}$, $n\in\mathbb{N}$, it follows that, in the limit as $n\to\infty$, $\mathrm{I}_V^e[\mu^e]\leqslant E_V^e := \inf\{\mathrm{I}_V^e[\mu^e];\ \mu^e\in\mathcal{M}_1(\mathbb{R})\}$; from the latter two inequalities, it follows, thus, that $\exists\ \mu^e := \mu_V^e\in\mathcal{M}_1(\mathbb{R})$, the 'even' equilibrium measure, such that $\mathrm{I}_V^e[\mu_V^e] = \inf\{\mathrm{I}_V^e[\mu^e];\ \mu^e\in\mathcal{M}_1(\mathbb{R})\}$, that is, the infimum is attained (the uniqueness of $\mu_V^e\in\mathcal{M}_1(\mathbb{R})$ is proven in Lemma 3.3 below).

The compactness of $\mathrm{supp}(\mu_V^e) =: J_e$ is now established: actually, the following proof is true for any $\mu\in\mathcal{M}_1(\mathbb{R})$ achieving the above minimum; in particular, for $\mu=\mu_V^e$. Without loss of generality,

therefore, let $\mu_w \in \mathcal{M}_1(\mathbb{R})$ be such that $\mathrm{I}_V^e[\mu_w] = E_V^e$, and let D be any proper subset of $\mathbb{R}$ for which $\mu_w(D) := \int_D \mathrm{d}\mu_w(s) > 0$. As in [91], set

$$\mu_w^{\varepsilon}(z) := (1+\varepsilon\mu_w(D))^{-1}(\mu_w(z)+\varepsilon(\mu_w \restriction_D)(z)), \quad \varepsilon \in (-1,1),$$

where $\mu_w \restriction_D$ denotes the restriction of μ_w to D (note, also, that $\mu_w^{\varepsilon} > 0$ and bounded, and $\int_{\mathbb{R}} \mathrm{d}\mu_w^{\varepsilon}(s) = 1$). Using the fact that $K_V^e(s,t) = K_V^e(t,s)$, one shows that

$$\begin{aligned}
\mathrm{I}_V^e[\mu_w^{\varepsilon}] &= \iint_{\mathbb{R}^2} K_V^e(s,t)\,\mathrm{d}\mu_w^{\varepsilon}(s)\,\mathrm{d}\mu_w^{\varepsilon}(t)\\
&= (1+\varepsilon\mu_w(D))^{-2} \iint_{\mathbb{R}^2} K_V^e(s,t)(\mathrm{d}\mu_w(s)+\varepsilon\mathrm{d}(\mu_w\restriction_D)(s))(\mathrm{d}\mu_w(t)+\varepsilon\mathrm{d}(\mu_w\restriction_D)(t))\\
&= (1+\varepsilon\mu_w(D))^{-2}\Bigg(\mathrm{I}_V^e[\mu_w]+2\varepsilon \iint_{\mathbb{R}^2} K_V^e(s,t)\,\mathrm{d}\mu_w(s)\,\mathrm{d}(\mu_w\restriction_D)(t)\\
&+ \varepsilon^2 \iint_{\mathbb{R}^2} K_V^e(s,t)\,\mathrm{d}(\mu_w\restriction_D)(t)\,\mathrm{d}(\mu_w\restriction_D)(s)\Bigg).
\end{aligned}$$

(Note that all of the above integrals are finite due to the argument at the beginning of the proof.) By the minimal property of $\mu_w \in \mathcal{M}_1(\mathbb{R})$, it follows that $\partial_{\varepsilon}\mathrm{I}_V^e[\mu_w^{\varepsilon}] = 0$, that is,

$$\iint_{\mathbb{R}^2} (K_V^e(s,t) - \mathrm{I}_V^e[\mu_w])\,\mathrm{d}\mu_w(s)\,\mathrm{d}(\mu_w\restriction_D)(t) = 0;$$

but, recalling that, with $\widehat{\psi}_V^e(z) := 2\widetilde{V}(z) - \ln(z^2+1) - \ln(z^{-2}+1)$, $K_V^e(t,s) \geqslant \frac{1}{2}\widehat{\psi}_V^e(s) + \frac{1}{2}\widehat{\psi}_V^e(t)$, it follows from the above that

$$\begin{aligned}
\iint_{\mathbb{R}^2} \mathrm{I}_V^e[\mu_w]\,\mathrm{d}\mu_w(s)\,\mathrm{d}(\mu_w\restriction_D)(t) \geqslant \iint_{\mathbb{R}^2}\big(\tfrac{1}{2}\widehat{\psi}_V^e(s)+\tfrac{1}{2}\widehat{\psi}_V^e(t)\big)\mathrm{d}\mu_w(s)\,\mathrm{d}(\mu_w\restriction_D)(t) \Rightarrow\\
0 \geqslant \iint_{\mathbb{R}^2}\big(\tfrac{1}{2}\widehat{\psi}_V^e(s)+\tfrac{1}{2}\widehat{\psi}_V^e(t)-\mathrm{I}_V^e[\mu_w]\big)\mathrm{d}\mu_w(s)\,\mathrm{d}(\mu_w\restriction_D)(t),
\end{aligned}$$

whence

$$\int_{\mathbb{R}}\left(\widehat{\psi}_V^e(t)+\left(\int_{\mathbb{R}}\widehat{\psi}_V^e(s)\,\mathrm{d}\mu_w(s)\right)-2\mathrm{I}_V^e[\mu_w]\right)\mathrm{d}(\mu_w\restriction_D)(t) \leqslant 0.$$

Recalling that

$$\widehat{\psi}_V^e(x) := 2\widetilde{V}(x) - \ln(x^2+1) - \ln(x^{-2}+1) = \begin{cases} +\infty, & |x| \to \infty,\\ +\infty, & |x| \to 0,\end{cases}$$

it follows that, $\exists\, T_m > 1$ such that

$$\widehat{\psi}_V^e(t) + \int_{\mathbb{R}}\widehat{\psi}_V^e(s)\,\mathrm{d}\mu_w(s) - 2\mathrm{I}_V^e[\mu_w] \geqslant 1 \quad \text{for} \quad t \in \left((-T_m, -T_m^{-1}) \cup (T_m^{-1}, T_m)\right)^c$$

(note, also, that $+\infty > \mathrm{I}_V^e[\mu_w] = \iint_{\mathbb{R}^2} K_V^e(t,s)\,\mathrm{d}\mu_w(t)\,\mathrm{d}\mu_w(s) = \int_{\mathbb{R}}\widehat{\psi}_V^e(\xi)\,\mathrm{d}\mu_w(\xi)$ = a finite real number). Hence, if D $(\subset \mathbb{R})$ is such that $D \subset (\{|x| \geqslant T_m\} \cup \{|x| \leqslant T_m^{-1}\})$, $T_m > 1$, it follows from the above calculations that

$$0 \geqslant \int_{\mathbb{R}}\left(\widehat{\psi}_V^e(t)+\left(\int_{\mathbb{R}}\widehat{\psi}_V^e(s)\,\mathrm{d}\mu_w(s)\right)-2\mathrm{I}_V^e[\mu_w]\right)\mathrm{d}(\mu_w\restriction_D)(t) \geqslant 1,$$

which is a contradiction; hence, $\mathrm{supp}(\mu_w) \subseteq [-T_m, -T_m^{-1}] \cup [T_m^{-1}, T_m]$, $T_m > 1$; in particular, $J_e := \mathrm{supp}(\mu_V^e) \subseteq [-T_m, -T_m^{-1}] \cup [T_m^{-1}, T_m]$, $T_m > 1$, which establishes the compactness of the support of the 'even' equilibrium measure $\mu_V^e \in \mathcal{M}_1(\mathbb{R})$. Furthermore, it is worth noting that, since $J_e := \mathrm{supp}(\mu_V^e) =$ compact $(\subseteq \overline{\mathbb{R}} \setminus \{0, \pm\infty\})$, and $\widetilde{V} \colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is real analytic on J_e,

$$+\infty > E_V^e\ (= \mathrm{I}_V^e[\mu_V^e]) \ \geqslant \iint_{\mathbb{R}^2} \ln\left(|s-t|^2|st|^{-1}w^e(s)w^e(t)\right)^{-1}\mathrm{d}\mu_V^e(s)\,\mathrm{d}\mu_V^e(t)$$

$$= \iint_{J_e^2} \ln\Big(|s-t|^2|st|^{-1}w^e(s)w^e(t)\Big)^{-1} \mathrm{d}\mu_V^e(s)\,\mathrm{d}\mu_V^e(t) > -\infty;$$

moreover, a straightforward consequence of the fact just established is that J_e has positive logarithmic capacity, that is, $\operatorname{cap}(J_e)=\exp(-E_V^e)>0$. □

Remark 3.1. It is important to note from the latter part of the proof of Lemma 3.1 that $J_e \not\supseteq \{0,\pm\infty\}$. This can also be seen as follows. For ε some arbitrarily fixed, sufficiently small positive real number and $\Sigma_\varepsilon := \{z;\ w^e(z) \geqslant \varepsilon\}$, if $(s,t) \notin \Sigma_\varepsilon \times \Sigma_\varepsilon$, then $\ln(|s-t|^2|st|^{-1}w^e(s)w^e(t))^{-1} =: K_V^e(s,t)$ $(= K_V^e(t,s)) > E_V^e+1$, which is a contradiction, since it was established above that the minimum is attained $\Leftrightarrow (s,t)\in\Sigma_\varepsilon\times\Sigma_\varepsilon$. Towards this end, it is enough to show that (see, for example, [55]), if $\{(s_n,t_n)\}_{n=1}^{\infty}$ is a sequence with $\lim\min_{n\to\infty}\{w^e(s_n),w^e(t_n)\}=0$, then $\lim_{n\to\infty}\ln(|s_n-t_n|^2|s_nt_n|^{-1}w^e(s_n)w^e(t_n))^{-1} = \lim_{n\to\infty}K_V^e(s_n,t_n)=+\infty$. Without loss of generality, one can assume that $s_n\to s$ and $t_n\to t$ as $n\to\infty$, where s, t, or both may be infinite; thus, there are several cases to consider:

(i) if s and t are finite, then, from $\lim\min_{n\to\infty}\{w^e(s_n),w^e(t_n)\}=\min\{w^e(s),w^e(t)\}=0$, it is clear that $\lim_{n\to\infty}K_V^e(s_n,t_n)=+\infty$;

(ii) if $|s|=\infty$ (resp., $|t|=\infty$) but $t=$ finite (resp., $s=$ finite), then, due to the fact that $\widetilde{V}\colon \mathbb{R}\setminus\{0\}\to\mathbb{R}$ satisfies the conditions

$$2\widetilde{V}(x)-\ln(x^2+1)-\ln(x^{-2}+1)=\begin{cases}+\infty, & |x|\to\infty,\\ +\infty, & |x|\to 0,\end{cases}$$

it follows that $\lim_{n\to\infty}K_V^e(s_n,t_n)=+\infty$;

(iii) if $|s|=0$ (resp., $|t|=0$) but $t=$ finite (resp., $s=$ finite), then, as a result of the above conditions for $\widetilde{V}$, it follows that $\lim_{n\to\infty}K_V^e(s_n,t_n)=+\infty$;

(iv) if $|s|=\infty$ and $|t|=\infty$, then, again due to the above conditions for $\widetilde{V}$, it follows that $\lim_{n\to\infty}K_V^e$ $(s_n,t_n)=+\infty$; and

(v) if $|s|=0$ and $|t|=0$, then, again, as above, it follows that $\lim_{n\to\infty}K_V^e(s_n,t_n)=+\infty$.

Hence, $K_V^e(s,t)>E_V^e+1$ if $(s,t)\notin\Sigma_\varepsilon\times\Sigma_\varepsilon$, that is, if s, t, or both $\in\{0,\pm\infty\}$ (which can not be the case, as the infimum E_V^e is attained $\Leftrightarrow (s,t)\in\Sigma_\varepsilon\times\Sigma_\varepsilon$, whence $\operatorname{supp}(\mu_V^e)=:J_e\not\supseteq\{0,\pm\infty\}$). ■

In order to establish the uniqueness of the 'even' equilibrium measure, μ_V^e $(\in\mathcal{M}_1(\mathbb{R}))$, the following lemma is requisite.

Lemma 3.2. *Let $\mu:=\mu_1-\mu_2$, where μ_1,μ_2 are non-negative, finite-moment ($\int_{\operatorname{supp}(\mu_j)}s^m\,\mathrm{d}\mu_j(s)<\infty$, $m\in\mathbb{Z}$, $j=1,2$) measures on $\mathbb{R}$ supported on distinct sets ($\operatorname{supp}(\mu_1)\cap\operatorname{supp}(\mu_2)=\varnothing$), be the (unique) Jordan decomposition of the finite-moment signed measure on $\mathbb{R}$ with mean zero, that is, $\int_{\operatorname{supp}(\mu)}\mathrm{d}\mu(s)=0$, and with $\operatorname{supp}(\mu)=$compact. Suppose that $-\infty<\iint_{\mathbb{R}^2}\ln(|s-t|^{-2}|st|)\,\mathrm{d}\mu_j(s)\,\mathrm{d}\mu_j(t)<+\infty$, $j=1,2$. Then,*

$$\iint_{\mathbb{R}^2}\ln\left(\frac{|st|}{|s-t|^2}\right)\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)=\iint_{\mathbb{R}^2}\ln\left(\frac{|s-t|^2}{|st|}w^e(s)w^e(t)\right)^{-1}\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)\geqslant 0,$$

where equality holds if, and only if, $\mu=0$.

Proof. Recall the following identity [90] (see pg. 147, Equation (6.44)): for $\xi\in\mathbb{R}$ and any $\varepsilon>0$,

$$\ln(\xi^2+\varepsilon^2)=\ln(\varepsilon^2)+2\operatorname{Im}\left(\int_0^{+\infty}\left(\frac{\mathrm{e}^{\mathrm{i}\xi v}-1}{\mathrm{i}v}\right)\mathrm{e}^{-\varepsilon v}\,\mathrm{d}v\right);$$

thus, it follows that

$$\begin{aligned}\iint_{\mathbb{R}^2}\ln((s-t)^2+\varepsilon^2)\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t) &= \iint_{\mathbb{R}^2}\ln(\varepsilon^2)\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)\\ &\quad+\iint_{\mathbb{R}^2}\left(2\operatorname{Im}\left(\int_0^{+\infty}\left(\frac{\mathrm{e}^{\mathrm{i}(s-t)v}-1}{\mathrm{i}v}\right)\mathrm{e}^{-\varepsilon v}\,\mathrm{d}v\right)\right)\mathrm{d}\mu(s)\,\mathrm{d}\mu(t),\\ \iint_{\mathbb{R}^2}\ln(s^2+\varepsilon^2)\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t) &= \iint_{\mathbb{R}^2}\ln(\varepsilon^2)\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)\end{aligned}$$

$$
\begin{aligned}
&\quad + \iint_{\mathbb{R}^2}\left(2\,\mathrm{Im}\left(\int_0^{+\infty}\left(\frac{\mathrm{e}^{\mathrm{i}sv}-1}{\mathrm{i}v}\right)\mathrm{e}^{-\varepsilon v}\,\mathrm{d}v\right)\right)\mathrm{d}\mu(s)\,\mathrm{d}\mu(t),\\
\iint_{\mathbb{R}^2}\ln(t^2+\varepsilon^2)\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t) &= \iint_{\mathbb{R}^2}\ln(\varepsilon^2)\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)\\
&\quad + \iint_{\mathbb{R}^2}\left(2\,\mathrm{Im}\left(\int_0^{+\infty}\left(\frac{\mathrm{e}^{\mathrm{i}tv}-1}{\mathrm{i}v}\right)\mathrm{e}^{-\varepsilon v}\,\mathrm{d}v\right)\right)\mathrm{d}\mu(s)\,\mathrm{d}\mu(t);
\end{aligned}
$$

but, since $\iint_{\mathbb{R}^2}\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)=\left(\int_{\mathbb{R}}\mathrm{d}\mu(s)\right)^2=0$, one obtains, after some rearrangement,

$$
\begin{aligned}
\iint_{\mathbb{R}^2}\ln((s-t)^2+\varepsilon^2)\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t) &= 2\,\mathrm{Im}\left(\int_0^{+\infty}\mathrm{e}^{-\varepsilon v}\left(\iint_{\mathbb{R}^2}\left(\frac{\mathrm{e}^{\mathrm{i}(s-t)v}-1}{\mathrm{i}v}\right)\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)\right)\mathrm{d}v\right),\\
\iint_{\mathbb{R}^2}\ln(s^2+\varepsilon^2)\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t) &= 2\,\mathrm{Im}\left(\int_0^{+\infty}\mathrm{e}^{-\varepsilon v}\left(\iint_{\mathbb{R}^2}\left(\frac{\mathrm{e}^{\mathrm{i}sv}-1}{\mathrm{i}v}\right)\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)\right)\mathrm{d}v\right),\\
\iint_{\mathbb{R}^2}\ln(t^2+\varepsilon^2)\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t) &= 2\,\mathrm{Im}\left(\int_0^{+\infty}\mathrm{e}^{-\varepsilon v}\left(\iint_{\mathbb{R}^2}\left(\frac{\mathrm{e}^{\mathrm{i}tv}-1}{\mathrm{i}v}\right)\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)\right)\mathrm{d}v\right).
\end{aligned}
$$

Noting that

$$
\begin{aligned}
\iint_{\mathbb{R}^2}\left(\frac{\mathrm{e}^{\mathrm{i}(s-t)v}-1}{\mathrm{i}v}\right)\mathrm{d}\mu(s)\,\mathrm{d}\mu(t) &= \frac{1}{\mathrm{i}v}\iint_{\mathbb{R}^2}\mathrm{e}^{\mathrm{i}(s-t)v}\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)-\frac{1}{\mathrm{i}v}\underbrace{\iint_{\mathbb{R}^2}\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)}_{=0}\\
&= \frac{1}{\mathrm{i}v}\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}sv}\,\mathrm{d}\mu(s)\int_{\mathbb{R}}\mathrm{e}^{-\mathrm{i}tv}\,\mathrm{d}\mu(t),
\end{aligned}
$$

and setting $\widehat{\mu}(z):=\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}\xi z}\,\mathrm{d}\mu(\xi)$, one gets that

$$
\iint_{\mathbb{R}^2}\left(\frac{\mathrm{e}^{\mathrm{i}(s-t)v}-1}{\mathrm{i}v}\right)\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)=\frac{1}{\mathrm{i}v}|\widehat{\mu}(v)|^2:
$$

also,

$$
\iint_{\mathbb{R}^2}\left(\frac{\mathrm{e}^{\mathrm{i}sv}-1}{\mathrm{i}v}\right)\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)=\frac{1}{\mathrm{i}v}\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}sv}\,\mathrm{d}\mu(s)\underbrace{\int_{\mathbb{R}}\mathrm{d}\mu(t)}_{=0}-\frac{1}{\mathrm{i}v}\underbrace{\int_{\mathbb{R}}\mathrm{d}\mu(s)}_{=0}\;\underbrace{\int_{\mathbb{R}}\mathrm{d}\mu(t)}_{=0}=0;
$$

similarly,

$$
\iint_{\mathbb{R}^2}\left(\frac{\mathrm{e}^{\mathrm{i}tv}-1}{\mathrm{i}v}\right)\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)=0.
$$

Hence,

$$
\begin{aligned}
&\iint_{\mathbb{R}^2}\ln((s-t)^2+\varepsilon^2)\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)=2\,\mathrm{Im}\left(\int_0^{+\infty}\frac{|\widehat{\mu}(v)|^2}{\mathrm{i}v}\mathrm{e}^{-\varepsilon v}\,\mathrm{d}v\right),\\
&\iint_{\mathbb{R}^2}\ln(s^2+\varepsilon^2)\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)=\iint_{\mathbb{R}^2}\ln(t^2+\varepsilon^2)\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)=0.
\end{aligned}
$$

Noting that $\widehat{\mu}(0)=\int_{\mathbb{R}}\mathrm{d}\mu(\xi)=0$, a Taylor expansion about $v=0$ shows that $\widehat{\mu}(v)=_{v\to 0}\widehat{\mu}'(0)v+O(v^2)$, where $\widehat{\mu}'(0):=\partial_v\widehat{\mu}(v)|_{v=0}$; thus, $v^{-1}|\widehat{\mu}(v)|^2=_{v\to 0}|\widehat{\mu}'(0)|^2v+O(v^2)$, which means that there is no singularity in the integrand as $v\to 0$ (in fact, $v^{-1}|\widehat{\mu}(v)|^2$ is real analytic in a neighbourhood of the origin), whence

$$
\iint_{\mathbb{R}^2}\ln((s-t)^2+\varepsilon^2)\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)=-2\int_0^{+\infty}v^{-1}|\widehat{\mu}(v)|^2\mathrm{e}^{-\varepsilon v}\,\mathrm{d}v.
$$

Recalling that $\iint_{\mathbb{R}^2}\ln(*^2+\varepsilon^2)\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)=0$, $*\in\{s,t\}$, and adding, it follows that

$$
\iint_{\mathbb{R}^2}\ln\left(\frac{(s^2+\varepsilon^2)^{1/2}(t^2+\varepsilon^2)^{1/2}}{((s-t)^2+\varepsilon^2)}\right)\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)=2\int_0^{+\infty}v^{-1}|\widehat{\mu}(v)|^2\mathrm{e}^{-\varepsilon v}\,\mathrm{d}v.
$$

Now, using the fact that $\ln((s-t)^2+\varepsilon^2)^{-1}$ (resp., $\ln(s^2+\varepsilon^2)^{1/2}$ and $\ln(t^2+\varepsilon^2)^{1/2}$) is (resp., are) bounded below (resp., above) uniformly with respect to ε and that the measures have compact support, letting $\varepsilon\downarrow 0$ and using the Monotone Convergence Theorem, one arrives at

$$\iint_{\mathbb{R}^2}\ln\left(\frac{(s^2+\varepsilon^2)^{1/2}(t^2+\varepsilon^2)^{1/2}}{((s-t)^2+\varepsilon^2)}\right)\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)\underset{\varepsilon\downarrow 0}{=}\iint_{\mathbb{R}^2}\ln\left(\frac{|st|}{|s-t|^2}\right)\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)$$
$$=2\int_0^{+\infty}v^{-1}|\widehat{\mu}(v)|^2\,\mathrm{d}v\geqslant 0,$$

where, trivially, equality holds if, and only if, $\mu=0$. Furthermore, noting that, since $\int_{\mathbb{R}}\mathrm{d}\mu(\xi)=0$, $\iint_{\mathbb{R}^2}\ln(w^e(*))^{-1}\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)=0$, $*\in\{s,t\}$, letting $\varepsilon\downarrow 0$ and using monotone convergence, one also arrives at

$$\iint_{\mathbb{R}^2}\ln\left(\frac{(s^2+\varepsilon^2)^{1/2}(t^2+\varepsilon^2)^{1/2}}{((s-t)^2+\varepsilon^2)w^e(s)w^e(t)}\right)\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)\underset{\varepsilon\downarrow 0}{=}\iint_{\mathbb{R}^2}\ln\left(\frac{|s-t|^2}{|st|}w^e(s)w^e(t)\right)^{-1}\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)$$
$$=2\int_0^{+\infty}v^{-1}|\widehat{\mu}(v)|^2\,\mathrm{d}v\geqslant 0,$$

where, again, and trivially, equality holds if, and only if, $\mu=0$. □

The uniqueness of μ_V^e $(\in\mathcal{M}_1(\mathbb{R}))$ will now be established.

Lemma 3.3. *Let the external field* $\widetilde{V}\colon\mathbb{R}\setminus\{0\}\to\mathbb{R}$ *satisfy conditions* (2.3)–(2.5)*. Set* $w^e(z):=\exp(-\widetilde{V}(z))$*, and define*

$$\mathrm{I}_V^e[\mu^e]\colon\ \mathcal{M}_1(\mathbb{R})\to\mathbb{R},\ \ \mu^e\mapsto\iint_{\mathbb{R}^2}\ln\Big(|s-t|^2|st|^{-1}w^e(s)w^e(t)\Big)^{-1}\,\mathrm{d}\mu^e(s)\,\mathrm{d}\mu^e(t),$$

and consider the minimisation problem $E_V^e=\inf\{\mathrm{I}_V^e[\mu^e];\ \mu^e\in\mathcal{M}_1(\mathbb{R})\}$*. Then,* $\exists!\ \mu_V^e\in\mathcal{M}_1(\mathbb{R})$ *such that* $\mathrm{I}_V^e[\mu_V^e]=E_V^e$*.*

Proof. It was shown in Lemma 3.1 that $\exists\ \mu_V^e\in\mathcal{M}_1(\mathbb{R})$, the 'even' equilibrium measure, such that $\mathrm{I}_V^e[\mu^e]=E_V^e$; therefore, it remains to establish the uniqueness of the 'even' equilibrium measure. Let $\widetilde{\mu}_V^e\in\mathcal{M}_1(\mathbb{R})$ be a second probability measure for which $\mathrm{I}_V^e[\widetilde{\mu}_V^e]=E_V^e=\mathrm{I}_V^e[\mu_V^e]$: the argument in Lemma 3.1 shows that $\widetilde{J}_e:=\operatorname{supp}(\widetilde{\mu}_V^e)=\text{compact}\subseteq\overline{\mathbb{R}}\setminus\{0,\pm\infty\}$, and that $\mathrm{I}_V^e[\widetilde{\mu}_V^e]<+\infty$. Define the finite-moment signed measure $\mu^\sharp:=\widetilde{\mu}_V^e-\mu_V^e$, where $\widetilde{\mu}_V^e,\mu_V^e\in\mathcal{M}_1(\mathbb{R})$, and $\widetilde{J}_e\cap J_e=\varnothing$, with (cf. Lemma 3.1), $J_e=\operatorname{supp}(\mu_V^e)=\text{compact}\subseteq\overline{\mathbb{R}}\setminus\{0,\pm\infty\}$; thus, from Lemma 3.2 (with $\mu\to\mu^\sharp$), namely,

$$\iint_{\mathbb{R}^2}\ln\Big(|s-t|^{-2}|st|\Big)\,\mathrm{d}\mu^\sharp(s)\,\mathrm{d}\mu^\sharp(t)=\iint_{\mathbb{R}^2}\ln\Big(|s-t|^2|st|^{-1}w^e(s)w^e(t)\Big)^{-1}\,\mathrm{d}\mu^\sharp(s)\,\mathrm{d}\mu^\sharp(t)\geqslant 0,$$

it follows that

$$\iint_{\mathbb{R}^2}\ln\Big(|st||s-t|^{-2}\Big)\Big(\mathrm{d}\widetilde{\mu}_V^e(s)\,\mathrm{d}\widetilde{\mu}_V^e(t)+\mathrm{d}\mu_V^e(s)\,\mathrm{d}\mu_V^e(t)\Big)\geqslant\iint_{\mathbb{R}^2}\ln\Big(|st||s-t|^{-2}\Big)\Big(\mathrm{d}\widetilde{\mu}_V^e(s)\,\mathrm{d}\mu_V^e(t)$$
$$+\ \mathrm{d}\mu_V^e(s)\mathrm{d}\widetilde{\mu}_V^e(t)\Big),$$

or, via a straightforward symmetry argument,

$$\iint_{\mathbb{R}^2}\ln\Big(|st||s-t|^{-2}\Big)\Big(\mathrm{d}\widetilde{\mu}_V^e(s)\,\mathrm{d}\widetilde{\mu}_V^e(t)+\mathrm{d}\mu_V^e(s)\,\mathrm{d}\mu_V^e(t)\Big)\geqslant 2\iint_{\mathbb{R}^2}\ln\Big(|st||s-t|^{-2}\Big)\mathrm{d}\widetilde{\mu}_V^e(s)\,\mathrm{d}\mu_V^e(t)$$
$$=2\iint_{\mathbb{R}^2}\ln\Big(|st||s-t|^{-2}\Big)\mathrm{d}\mu_V^e(s)\,\mathrm{d}\widetilde{\mu}_V^e(t).$$

The above shows that (since both $\mathrm{I}_V^e[\mu_V^e]$ and $\mathrm{I}_V^e[\widetilde{\mu}_V^e]<+\infty$) $\ln(|st||s-t|^{-2})$ is integrable with respect to both $\mathrm{d}\widetilde{\mu}_V^e(s)\,\mathrm{d}\mu_V^e(t)$ and $\mathrm{d}\mu_V^e(s)\,\mathrm{d}\widetilde{\mu}_V^e(t)$. From an argument on pg. 149 of [90], it follows that $\ln(|st||s-t|^{-2})$ is integrable with respect to (the measure) $\mathrm{d}\mu_t^e(s)\,\mathrm{d}\mu_t^e(t')$, where $\mu_t^e(z):=\mu_V^e(z)+t(\widetilde{\mu}_V^e(z)-\mu_V^e(z))$, $(z,t)\in\mathbb{R}\times[0,1]$. Set

$$\mathcal{F}_\mu(t):=\iint_{\mathbb{R}^2}\ln\Big(|st'||s-t'|^{-2}(w^e(s)w^e(t'))^{-1}\Big)\,\mathrm{d}\mu_t^e(s)\,\mathrm{d}\mu_t^e(t')$$

($=\mathrm{I}_V^e[\mu_t^e]$). Noting that

$$\begin{aligned}\mathrm{d}\mu_t^e(s)\,\mathrm{d}\mu_t^e(t') &= \mathrm{d}\mu_V^e(s)\,\mathrm{d}\mu_V^e(t')+t\mathrm{d}\mu_V^e(s)(\mathrm{d}\widetilde{\mu}_V^e(t')-\mathrm{d}\mu_V^e(t'))+t\mathrm{d}\mu_V^e(t')(\mathrm{d}\widetilde{\mu}_V^e(s)-\mathrm{d}\mu_V^e(s))\\ &\quad+t^2(\mathrm{d}\widetilde{\mu}_V^e(s)-\mathrm{d}\mu_V^e(s))(\mathrm{d}\widetilde{\mu}_V^e(t')-\mathrm{d}\mu_V^e(t')),\end{aligned}$$

it follows that

$$\begin{aligned}\mathcal{F}_\mu(t) &= \mathrm{I}_V^e[\mu_V^e]+2t\iint_{\mathbb{R}^2}\ln\left(\frac{|st'|}{|s-t'|^2}(w^e(s)w^e(t'))^{-1}\right)\mathrm{d}\mu_V^e(s)(\mathrm{d}\widetilde{\mu}_V^e(t')-\mathrm{d}\mu_V^e(t'))\\ &\quad+t^2\iint_{\mathbb{R}^2}\ln\left(\frac{|st'|}{|s-t'|^2}(w^e(s)w^e(t'))^{-1}\right)(\mathrm{d}\widetilde{\mu}_V^e(s)-\mathrm{d}\mu_V^e(s))(\mathrm{d}\widetilde{\mu}_V^e(t')-\mathrm{d}\mu_V^e(t')).\end{aligned}$$

Since $\mu^\sharp\in\mathcal{M}_1(\mathbb{R})$ is a finite-moment signed measure with mean zero, that is, $\int_{\mathbb{R}}\mathrm{d}\mu^\sharp(\xi)=\int_{\mathbb{R}}\mathrm{d}(\widetilde{\mu}_V^e-\mu_V^e)(\xi)=0$, and compact support, it follows from the analysis above and the result of Lemma 3.2 that $\mathcal{F}_\mu(t)$ is convex[9]; thus, for $t\in[0,1]$,

$$\begin{aligned}\mathrm{I}_V^e[\mu_V^e] &\leqslant \mathcal{F}_\mu(t)=\mathrm{I}_V^e[\mu_t^e]=\mathcal{F}_\mu(t+(1-t)0)\leqslant t\mathcal{F}_\mu(1)+(1-t)\mathcal{F}_\mu(0)\\ &= t\mathrm{I}_V^e[\widetilde{\mu}_V^e]+(1-t)\mathrm{I}_V^e[\mu_V^e]=t\mathrm{I}_V^e[\mu_V^e]+(1-t)\mathrm{I}_V^e[\mu_V^e]\Rightarrow\\ \mathrm{I}_V^e[\mu_V^e] &\leqslant \mathrm{I}_V^e[\mu_t^e]\leqslant\mathrm{I}_V^e[\mu_V^e],\end{aligned}$$

whence $\mathrm{I}_V^e[\mu_t^e]=\mathrm{I}_V^e[\mu_V^e]:=E_V^e$ (= const.). Since $\mathrm{I}_V^e[\mu_t^e]=\mathcal{F}_\mu(t)=E_V^e$, it follows, in particular, that $\mathcal{F}_\mu''(0)=0\Rightarrow$

$$\begin{aligned}0 &= \iint_{\mathbb{R}^2}\ln\left(\frac{|st'|}{|s-t'|^2}(w^e(s)w^e(t'))^{-1}\right)(\mathrm{d}\widetilde{\mu}_V^e(s)-\mathrm{d}\mu_V^e(s))(\mathrm{d}\widetilde{\mu}_V^e(t')-\mathrm{d}\mu_V^e(t'))\\ &= \iint_{\mathbb{R}^2}\ln\left(\frac{|st'|}{|s-t'|^2}\right)(\mathrm{d}\widetilde{\mu}_V^e(s)-\mathrm{d}\mu_V^e(s))(\mathrm{d}\widetilde{\mu}_V^e(t')-\mathrm{d}\mu_V^e(t'))\\ &\quad+2\int_{\mathbb{R}}\widetilde{V}(t')\,\mathrm{d}(\widetilde{\mu}_V^e-\mu_V^e)(t')\underbrace{\int_{\mathbb{R}}\mathrm{d}(\widetilde{\mu}_V^e-\mu_V^e)(s)}_{=0}\Rightarrow\\ 0 &= \iint_{\mathbb{R}^2}\ln\left(\frac{|st'|}{|s-t'|^2}\right)\mathrm{d}(\widetilde{\mu}_V^e-\mu_V^e)(s)\,\mathrm{d}(\widetilde{\mu}_V^e-\mu_V^e)(t');\end{aligned}$$

but, in Lemma 3.2, it was shown that

$$\iint_{\mathbb{R}^2}\ln\left(\frac{|st'|}{|s-t'|^2}\right)\mathrm{d}(\widetilde{\mu}_V^e-\mu_V^e)(s)\,\mathrm{d}(\widetilde{\mu}_V^e-\mu_V^e)(t')=2\int_0^{+\infty}\xi^{-1}|(\widehat{\widetilde{\mu}_V^e}-\widehat{\mu_V^e})(\xi)|^2\,\mathrm{d}\xi\geqslant0,$$

whence $\int_0^{+\infty}\xi^{-1}|(\widehat{\widetilde{\mu}_V^e}-\widehat{\mu_V^e})(\xi)|^2\,\mathrm{d}\xi=0\Rightarrow\widehat{\widetilde{\mu}_V^e}(\xi)=\widehat{\mu_V^e}(\xi)$, $\xi\geqslant0$. Noting that

$$\widehat{\widetilde{\mu}_V^e}(-\xi)=\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}s(-\xi)}\,\mathrm{d}\widetilde{\mu}_V^e(s)=\overline{\widehat{\widetilde{\mu}_V^e}(\xi)}\qquad\text{and}\qquad\widehat{\mu_V^e}(-\xi)=\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}s(-\xi)}\,\mathrm{d}\mu_V^e(s)=\overline{\widehat{\mu_V^e}(\xi)},$$

it follows from $\widehat{\widetilde{\mu}_V^e}(\xi)=\widehat{\mu_V^e}(\xi)$, $\xi\geqslant0$, via a complex-conjugation argument, that $\widehat{\widetilde{\mu}_V^e}(-\xi)=\widehat{\mu_V^e}(-\xi)$, $\xi\geqslant0$; hence, $\widehat{\widetilde{\mu}_V^e}(\xi)=\widehat{\mu_V^e}(\xi)$, $\xi\in\mathbb{R}$. The latter relation shows that $\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}s\xi}\,\mathrm{d}(\widetilde{\mu}_V^e-\mu_V^e)(s)=0\Rightarrow\widetilde{\mu}_V^e=\mu_V^e$; thus the uniqueness of the 'even' equilibrium measure. □

Before proceeding to Lemma 3.4, the following observations, which are interesting, non-trivial and important results in their own right, should be noted. Let $\widetilde{V}\colon\mathbb{R}\setminus\{0\}\to\mathbb{R}$ satisfy conditions (2.3)–(2.5). For each $n\in\mathbb{N}$ and any $2n$-tuple $(x_1,x_2,\dots,x_{2n})$ of distinct, finite and non-zero real numbers, let

$$\mathfrak{d}_{e,n}^{\widetilde{V}}:=\frac{1}{2n(2n-1)}\inf_{\{x_1,x_2,\dots,x_{2n}\}\subset\mathbb{R}\setminus\{0\}}\left(\sum_{\substack{j,k=1\\j\neq k}}^{2n}\ln\left(|x_j-x_k||x_k^{-1}-x_j^{-1}|\right)^{-1}+2(2n-1)\sum_{i=1}^{2n}\widetilde{V}(x_i)\right).$$

[9]If f is twice differentiable on (a,b), then $f''(x)\geqslant0$ on (a,b) is both a necessary and sufficient condition that f be convex on (a,b).

For each $n \in \mathbb{N}$, a set $\left\{x_1^\sharp, x_2^\sharp, \dots, x_{2n}^\sharp\right\}$ which realizes the above infimum, that is,

$$\mathfrak{d}_{e,n}^{\widetilde{V}} = \frac{1}{2n(2n-1)} \left(\sum_{\substack{j,k=1 \\ j \neq k}}^{2n} \ln\left(\left| x_j^\sharp - x_k^\sharp \right| \left| (x_k^\sharp)^{-1} - (x_j^\sharp)^{-1} \right| \right)^{-1} + 2(2n-1) \sum_{i=1}^{2n} \widetilde{V}(x_i^\sharp) \right),$$

will be called (with slight abuse of nomenclature) a *generalised weighted* $2n$*-Fekete set*, and the points $x_1^\sharp, x_2^\sharp, \dots, x_{2n}^\sharp$ will be called *generalised weighted Fekete points*. For $\left\{x_1^\sharp, x_2^\sharp, \dots, x_{2n}^\sharp\right\}$ a generalised weighted $2n$-Fekete set, denote by

$$\mu_{\mathbf{x}^\sharp}^e := \frac{1}{2n} \sum_{j=1}^{2n} \delta_{x_j^\sharp},$$

where $\delta_{x_j^\sharp}$, $j = 1, \dots, 2n$, is the Dirac delta measure (atomic mass) concentrated at $x_j^\sharp$, the *normalised counting measure*, that is, $\int_{\mathbb{R}} \mathrm{d}\mu_{\mathbf{x}^\sharp}^e(s) = 1$. Then, mimicking the calculations in Chapter 6 of [90] and the techniques used to prove Theorem 1.34 in [56] (see, in particular, Section 2 of [56]), one proves that (the details are left to the interested reader):

- $\lim_{n\to\infty} \mathfrak{d}_{e,n}^{\widetilde{V}}$ exists, more precisely,

$$\lim_{n\to\infty} \mathfrak{d}_{e,n}^{\widetilde{V}} = E_V^e = \inf\{\mathrm{I}_V^e[\mu^e];\ \mu^e \in \mathcal{M}_1(\mathbb{R})\},$$

  where (the functional) $\mathrm{I}_V^e[\mu^e] \colon \mathcal{M}_1(\mathbb{R}) \to \mathbb{R}$ is defined in Lemma 3.1, and $\lim_{n\to\infty} \exp(-\mathfrak{d}_{e,n}^{\widetilde{V}}) = \exp(-E_V^e)$ is positive and finite;
- $\mu_{\mathbf{x}^\sharp}^e$ converges weakly (in the weak-$*$ topology of measures) to the 'even' equilibrium measure μ_V^e, that is, $\mu_{\mathbf{x}^\sharp}^e \overset{*}{\to} \mu_V^e$ as $n \to \infty$.

RHP1, that is, $(\overset{e}{\mathrm{Y}}(z), \mathrm{I} + \exp(-n\widetilde{V}(z))\sigma_+, \mathbb{R})$, is now reformulated as an equivalent, auxiliary RHP normalised at infinity.

Notational Remark 3.1. For completeness, the integrand appearing in the definition of $g^e(z)$ (see Lemma 3.4 below) is defined as follows: $\ln((z-s)^2(zs)^{-1}) := 2\ln(z-s) - \ln z - \ln s$, where, for $s < 0$, $\ln s := \ln|s| + \mathrm{i}\pi$. ■

Lemma 3.4. *Let the external field* $\widetilde{V} \colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ *satisfy conditions* (2.3)–(2.5)*. For the associated 'even' equilibrium measure,* $\mu_V^e \in \mathcal{M}_1(\mathbb{R})$*, set* $J_e := \operatorname{supp}(\mu_V^e)$*, where* J_e *(= compact)* $\subset \overline{\mathbb{R}} \setminus \{0, \pm\infty\}$*, and let* $\overset{e}{\mathrm{Y}} \colon \mathbb{C} \setminus \mathbb{R} \to \mathrm{SL}_2(\mathbb{C})$ *be the (unique) solution of* **RHP1***. Let*

$$\overset{e}{\mathcal{M}}(z) := \mathrm{e}^{-\frac{n\ell_e}{2}\mathrm{ad}(\sigma_3)} \overset{e}{\mathrm{Y}}(z) \mathrm{e}^{-n(g^e(z) + Q_e)\sigma_3},$$

where $g^e(z)$*, the 'even'* g*-function, is defined by*

$$g^e(z) := \int_{J_e} \ln\left((z-s)^2(zs)^{-1}\right) \mathrm{d}\mu_V^e(s), \quad z \in \mathbb{C} \setminus (-\infty, \max\{0, \max\{\operatorname{supp}(\mu_V^e)\}\}),$$

ℓ_e $(\in \mathbb{R})$*, the 'even' variational constant, is given in* Lemma 3.6 *below, and*

$$Q_e := \int_{J_e} \ln(s) \mathrm{d}\mu_V^e(s).$$

Then $\overset{e}{\mathcal{M}} \colon \mathbb{C} \setminus \mathbb{R} \to \mathrm{SL}_2(\mathbb{C})$ *solves the following (normalised at infinity)* RHP: (i) $\overset{e}{\mathcal{M}}(z)$ *is holomorphic for* $z \in \mathbb{C} \setminus \mathbb{R}$; (ii) *the boundary values* $\overset{e}{\mathcal{M}}_\pm(z) := \lim_{\substack{z' \to z \\ \pm\mathrm{Im}(z') > 0}} \overset{e}{\mathcal{M}}(z')$ *satisfy the jump condition*

$$\overset{e}{\mathcal{M}}_+(z) = \overset{e}{\mathcal{M}}_-(z) \begin{pmatrix} \mathrm{e}^{-n(g_+^e(z) - g_-^e(z))} & \mathrm{e}^{n(g_+^e(z) + g_-^e(z) - \widetilde{V}(z) - \ell_e + 2Q_e)} \\ 0 & \mathrm{e}^{n(g_+^e(z) - g_-^e(z))} \end{pmatrix}, \quad z \in \mathbb{R},$$

where $g_\pm^e(z) := \lim_{\varepsilon \downarrow 0} g^e(z \pm \mathrm{i}\varepsilon)$; (iii) $\overset{e}{\mathcal{M}}(z) \underset{\substack{z\to\infty \\ z \in \mathbb{C}\setminus\mathbb{R}}}{=} \mathrm{I} + \mathcal{O}(z^{-1})$; *and* (iv) $\overset{e}{\mathcal{M}}(z) \underset{\substack{z\to 0 \\ z \in \mathbb{C}\setminus\mathbb{R}}}{=} \mathcal{O}(1)$.

Proof. For (arbitrary) $z_1, z_2 \in \mathbb{C}_\pm$, note that, from the definition of $g^e(z)$ stated in the Lemma, $g^e(z_2)-g^e(z_1)=\mathrm{i}\pi\int_{z_1}^{z_2}\mathcal{F}^e(s)\,\mathrm{d}s$, where

$$\mathcal{F}^e\colon \mathbb{C}\setminus(\operatorname{supp}(\mu_V^e)\cup\{0\})\to\mathbb{C},\; z\mapsto -\frac{1}{\pi\mathrm{i}}\left(\frac{1}{z}+2\int_{J_e}\frac{\mathrm{d}\mu_V^e(s)}{s-z}\right),$$

with $\mathcal{F}^e(z)=_{z\to 0}-\frac{1}{\pi\mathrm{i}z}+\mathcal{O}(1)$ (since $\mu_V^e\in\mathcal{M}_1(\mathbb{R})$; in particular, $\int_{\mathbb{R}}s^m\,\mathrm{d}\mu_V^e(s)<\infty$, $m\in\mathbb{Z}$); thus, $|g^e(z_2)-g^e(z_1)|\leqslant \pi\sup_{z\in\mathbb{C}_\pm}|\mathcal{F}^e(z)||z_2-z_1|$, that is, $g^e(z)$ is uniformly Lipschitz continuous in $\mathbb{C}_\pm$. Thus, from the definition of $g^e(z)$ stated in the Lemma:

(1) for $s\in J_e$, $z\in\mathbb{C}\setminus(-\infty,\max\{0,\max\{\operatorname{supp}(\mu_V^e)\}\})$, with $|s/z|\ll 1$, and $\mu_V^e\in\mathcal{M}_1(\mathbb{R})$, in particular, $\int_{\mathbb{R}}\mathrm{d}\mu_V^e(s)$ $(=\int_{J_e}\mathrm{d}\mu_V^e(s))=1$ and $\int_{\mathbb{R}}s^m\,\mathrm{d}\mu_V^e(s)$ $(=\int_{J_e}s^m\,\mathrm{d}\mu_V^e(s))<\infty$, $m\in\mathbb{N}$, it follows from the expansions $\frac{1}{s-z}=-\sum_{k=0}^{l}\frac{s^k}{z^{k+1}}+\frac{s^{l+1}}{z^{l+1}(s-z)}$, $l\in\mathbb{Z}_0^+$, and $\ln(z-s)=_{|z|\to\infty}\ln(z)-\sum_{k=1}^{\infty}\frac{1}{k}(\frac{s}{z})^k$, that

$$g^e(z)\underset{\mathbb{C}\setminus\mathbb{R}\ni z\to\infty}{=}\ln(z)-Q_e+\mathcal{O}(z^{-1}),$$

where Q_e is defined in the Lemma;

(2) for $s\in J_e$, $z\in\mathbb{C}\setminus(-\infty,\max\{0,\max\{\operatorname{supp}(\mu_V^e)\}\})$, with $|z/s|\ll 1$, and $\mu_V^e\in\mathcal{M}_1(\mathbb{R})$, in particular, $\int_{\mathbb{R}}s^{-m}\,\mathrm{d}\mu_V^e(s)$ $(=\int_{J_e}s^{-m}\,\mathrm{d}\mu_V^e(s))<\infty$, $m\in\mathbb{N}$, it follows from the expansions $\frac{1}{z-s}=-\sum_{k=0}^{l}\frac{z^k}{s^{k+1}}+\frac{z^{l+1}}{s^{l+1}(z-s)}$, $l\in\mathbb{Z}_0^+$, and $\ln(s-z)=_{|z|\to 0}\ln(s)-\sum_{k=1}^{\infty}\frac{1}{k}(\frac{z}{s})^k$, that

$$g^e(z)\underset{\mathbb{C}_\pm\ni z\to 0}{=}-\ln(z)-Q_e+2\int_{J_e}\ln(|s|)\,\mathrm{d}\mu_V^e(s)\pm 2\pi\mathrm{i}\int_{J_e\cap\mathbb{R}_+}\mathrm{d}\mu_V^e(s)+\mathcal{O}(z),$$

where (see Lemma 3.5, item (1), below)

$$\int_{J_e\cap\mathbb{R}_+}\mathrm{d}\mu_V^e(s)=\begin{cases}0, & J_e\subset\mathbb{R}_-,\\ 1, & J_e\subset\mathbb{R}_+,\\ \int_{b_j^e}^{a_{N+1}^e}\mathrm{d}\mu_V^e(s), & (a_j^e,b_j^e)\ni 0,\quad j=1,\dots,N.\end{cases}$$

Items (i)–(iv) now follow from the definitions of $\overset{e}{\mathcal{M}}(z)$ (in terms of $\overset{e}{\mathrm{Y}}(z)$) and $g^e(z)$ stated in the Lemma, and the above two asymptotic expansions. □

Lemma 3.5. *Let the external field $\widetilde{V}\colon\mathbb{R}\setminus\{0\}\to\mathbb{R}$ satisfy conditions* (2.3)–(2.5). *For $\mu_V^e\in\mathcal{M}_1(\mathbb{R})$, the associated 'even' equilibrium measure, set $J_e:=\operatorname{supp}(\mu_V^e)$, where J_e (= compact) $\subset\overline{\mathbb{R}}\setminus\{0,\pm\infty\}$. Then:* (1) *$J_e=\cup_{j=1}^{N+1}(b_{j-1}^e,a_j^e)$, with $N\in\mathbb{N}$ and finite, $b_0^e:=\min\{\operatorname{supp}(\mu_V^e)\}\notin\{-\infty,0\}$, $a_{N+1}^e:=\max\{\operatorname{supp}(\mu_V^e)\}\notin\{0,+\infty\}$, and $-\infty<b_0^e<a_1^e<b_1^e<a_2^e<\cdots<b_N^e<a_{N+1}^e<+\infty$, and $\{b_{j-1}^e,a_j^e\}_{j=1}^{N+1}$ satisfy the n-dependent and (locally) solvable system of* $2(N+1)$ *moment conditions*

$$\int_{J_e}\frac{(\frac{2\mathrm{i}}{\pi s}+\frac{\mathrm{i}\widetilde{V}'(s)}{\pi})s^j}{(R_e(s))_+^{1/2}}\frac{\mathrm{d}s}{2\pi\mathrm{i}}=0,\quad j=0,\dots,N,\qquad \int_{J_e}\frac{(\frac{2\mathrm{i}}{\pi s}+\frac{\mathrm{i}\widetilde{V}'(s)}{\pi})s^{N+1}}{(R_e(s))_+^{1/2}}\frac{\mathrm{d}s}{2\pi\mathrm{i}}=\frac{2}{\pi\mathrm{i}},$$

$$\int_{a_j^e}^{b_j^e}\left(\mathrm{i}(R_e(s))^{1/2}\int_{J_e}\frac{(\frac{\mathrm{i}}{\pi\xi}+\frac{\mathrm{i}\widetilde{V}'(\xi)}{2\pi})}{(R_e(\xi))_+^{1/2}(\xi-s)}\frac{\mathrm{d}\xi}{2\pi\mathrm{i}}\right)\mathrm{d}s=\frac{1}{2\pi}\ln\left|\frac{a_j^e}{b_j^e}\right|+\frac{1}{4\pi}\left(\widetilde{V}(a_j^e)-\widetilde{V}(b_j^e)\right),\quad j=1,\dots,N,$$

where $(R_e(z))^{1/2}$ is defined in Theorem 2.3.1, *Equation* (2.8), *with $(R_e(z))_\pm^{1/2}:=\lim_{\varepsilon\downarrow 0}(R_e(z\pm\mathrm{i}\varepsilon))^{1/2}$, and the branch of the square root chosen so that $z^{-(N+1)}(R_e(z))^{1/2}\sim_{\substack{z\to\infty\\ z\in\mathbb{C}_\pm}}\pm 1$; and* (2) *the density of the 'even' equilibrium measure, which is absolutely continuous with respect to Lebesgue measure, is given by*

$$\mathrm{d}\mu_V^e(x):=\psi_V^e(x)\,\mathrm{d}x=\frac{1}{2\pi\mathrm{i}}(R_e(x))_+^{1/2}h_V^e(x)\mathbf{1}_{J_e}(x)\,\mathrm{d}x,$$

where

$$h_V^e(z):=\frac{1}{2}\oint_{C_{\mathrm{R}}^e}\frac{(\frac{\mathrm{i}}{\pi s}+\frac{\mathrm{i}\widetilde{V}'(s)}{2\pi})}{(R_e(s))^{1/2}(s-z)}\,\mathrm{d}s$$

(real analytic for $z \in \mathbb{R} \setminus \{0\}$), with $C^{e}_{\mathbb{R}}$ ($\subset \mathbb{C}^{*}$) the boundary of any open doubly-connected annular region of the type $\{z' \in \mathbb{C};\ 0<r<|z'|<R<+\infty\}$, where the simple outer (resp., inner) boundary $\{z'=R\mathrm{e}^{\mathrm{i}\vartheta},\ 0 \leqslant \vartheta \leqslant 2\pi\}$ (resp., $\{z'=r\mathrm{e}^{\mathrm{i}\vartheta},\ 0 \leqslant \vartheta \leqslant 2\pi\}$) is traversed clockwise (resp., counter-clockwise), with the numbers $0<r<R<+\infty$ chosen such that, for (any) non-real z in the domain of analyticity of $\widetilde{V}$ (that is, $\mathbb{C}^{*}$), $\mathrm{int}(C^{e}_{\mathbb{R}}) \supset J_e \cup \{z\}$, $\mathbf{1}_{J_e}(x)$ is the indicator (characteristic) function of the set J_e, and $\psi^{e}_{V}(x) \geqslant 0$ (resp., $\psi^{e}_{V}(x)>0$) $\forall\ x \in \overline{J_e} := \cup_{j=1}^{N+1}[b^{e}_{j-1},a^{e}_{j}]$ (resp., $\forall\ x \in J_e$).

Proof. One begins by showing that the support of the 'even' equilibrium measure, $\mathrm{supp}(\mu^{e}_{V}) =: J_e$, consists of the union of a finite number of disjoint and bounded (real) intervals. Recall from Lemma 3.1 that $J_e = \text{compact} \subset \overline{\mathbb{R}} \setminus \{0,\pm\infty\}$, and that $\widetilde{V}$ is real analytic on $\mathbb{R} \setminus \{0\}$, thus real analytic on J_e, with an analytic continuation to the following (open) neighbourhood of J_e, $\mathbb{U} := \{z \in \mathbb{C};\ \inf_{q \in J_e}|z-q| < r \in (0,1)\} \setminus \{0\}$. In analogy with Equation (2.1) of [56], for each $n \in \mathbb{N}$ and any $2n$-tuple $(x_1,x_2,\dots,x_{2n})$ of distinct, finite and non-zero real numbers, let

$$
\begin{aligned}
\mathrm{d}^{e}_{\widetilde{V},n} &:= \left(\sup_{\{x_1,x_2,\dots,x_{2n}\} \subset \mathbb{R}\setminus\{0\}} \prod_{\substack{j,k=1\\ j<k}}^{2n} \left|x_j-x_k\right|^2 \left|x_k^{-1}-x_j^{-1}\right|^2 \mathrm{e}^{-2\widetilde{V}(x_j)}\mathrm{e}^{-2\widetilde{V}(x_k)} \right)^{\frac{1}{2n(2n-1)}} \\
&= \left(\sup_{\{x_1,x_2,\dots,x_{2n}\} \subset \mathbb{R}\setminus\{0\}} \prod_{\substack{j,k=1\\ j<k}}^{2n} \left|x_j-x_k\right|^2 \left|x_k^{-1}-x_j^{-1}\right|^2 \mathrm{e}^{-2(2n-1)\sum_{i=1}^{2n}\widetilde{V}(x_i)} \right)^{\frac{1}{2n(2n-1)}},
\end{aligned}
$$

where $\prod_{\substack{j,k=1\\ j<k}}^{2n}(\star) = \prod_{j=1}^{2n-1}\prod_{k=j+1}^{2n}(\star)$. Denote by $\left\{x_1^{*},x_2^{*},\dots,x_{2n}^{*}\right\}$, with $x_i^{*} < x_j^{*}\ \forall\ i<j \in \{1,\dots,2n\}$, the associated generalised weighted $2n$-Fekete set (see the discussion preceding Lemma 3.4), that is,

$$
\mathrm{d}^{e}_{\widetilde{V},n} = \left(\prod_{\substack{j,k=1\\ j<k}}^{2n} \left|x_j^{*}-x_k^{*}\right|^2 \left|(x_k^{*})^{-1}-(x_j^{*})^{-1}\right|^2 \mathrm{e}^{-2(2n-1)\sum_{i=1}^{2n}\widetilde{V}(x_i^{*})} \right)^{\frac{1}{2n(2n-1)}}.
$$

Proceeding, now, as in the proof of Theorem 1.34, Equation (1.35), of [56], in particular, mimicking the calculations on pp. 408–413 of [56] (for the proofs of Lemmas 2.3 and 2.15 therein), namely, using those techniques to show that, in the present case, the nearest-neighbour distances $\{x_{j+1}^{*}-x_j^{*}\}_{j=1}^{2n-1}$ are not 'too small' as $n \to \infty$, and the calculations on pp. 413–415 of [56] (for the proof of Lemma 2.26 therein), one shows that, for the regular case considered herein (cf. Subsection 2.2), the 'even' equilibrium measure, μ^{e}_{V} ($\in \mathcal{M}_1(\mathbb{R})$), is absolutely continuous with respect to Lebesgue measure, that is, the density of the 'even' equilibrium measure has the representation $\mathrm{d}\mu^{e}_{V}(x) := \psi^{e}_{V}(x)\,\mathrm{d}x$, $x \in \mathrm{supp}(\mu^{e}_{V})$, where $\psi^{e}_{V}(x) \geqslant 0$ on $\overline{J_e}$, with $\psi^{e}_{V}(\cdot)$ determined (explicitly) below[10].

Set

$$
\mathcal{H}^{e}(z) := (\mathcal{F}^{e}(z))^2 - \int_{J_e} \frac{(16\mathrm{i}\psi^{e}_{V}(\xi)(\mathcal{H}\psi^{e}_{V})(\xi) - 8\mathrm{i}\psi^{e}_{V}(\xi)/\pi\xi)}{(\xi-z)} \frac{\mathrm{d}\xi}{2\pi\mathrm{i}}, \quad z \in \mathbb{C} \setminus (J_e \cup \{0\}), \tag{3.1}
$$

where, from the proof of Lemma 3.4,

$$
\mathcal{F}^{e}(z) = -\frac{1}{\pi\mathrm{i}}\left(\frac{1}{z} + 2\int_{J_e} \frac{\mathrm{d}\mu^{e}_{V}(s)}{s-z}\right), \tag{3.2}
$$

with $\int_{J_e} \frac{\mathrm{d}\mu^{e}_{V}(s)}{s-z}$ the Stieltjes transform of the 'even' equilibrium measure, and

$$
\mathcal{H} \colon \mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})} \to \mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})},\ f \mapsto (\mathcal{H}f)(z) := \fint_{\mathbb{R}} \frac{f(s)}{z-s}\frac{\mathrm{d}s}{\pi}
$$

[10]The analysis of [56] is, in some sense, more complicated than the one of the present paper, because, unlike the 'real-line' case considered herein, that is, $\mathrm{supp}(\mu^{e}_{V}) =: J_e \subset \overline{\mathbb{R}} \setminus \{0,\pm\infty\}$, the end-point effects at ± 1 in [56] require special consideration (see, also, Section 4 of [56]).

denotes the Hilbert transform, with $⨍$ denoting the principle value integral. Via the distributional identities $\frac{1}{x-(x_0\pm \mathrm{i}0)} = \frac{1}{x-x_0} \pm \mathrm{i}\pi\delta(x-x_0)$, with $\delta(\cdot)$ the Dirac delta function, and $\int_{\xi_1}^{\xi_2} f(\xi)\delta(\xi-x)\mathrm{d}\xi = \begin{cases} f(x), & x\in(\xi_1,\xi_2), \\ 0, & x\in\mathbb{R}\setminus(\xi_1,\xi_2), \end{cases}$ it follows that

$$\mathcal{H}^e_{\pm}(z)= \begin{cases} (\mathcal{F}^e_{\pm}(z))^2 - ⨍_{J_e} \dfrac{(16\mathrm{i}\psi^e_V(\xi)(\mathcal{H}\psi^e_V)(\xi)-\frac{8\mathrm{i}\psi^e_V(\xi)}{\pi\xi})}{(\xi-z)}\dfrac{\mathrm{d}\xi}{2\pi\mathrm{i}} \mp \dfrac{1}{2}\left(16\mathrm{i}\psi^e_V(z)(\mathcal{H}\psi^e_V)(z)-\dfrac{8\mathrm{i}\psi^e_V(z)}{\pi z}\right), & z\in J_e, \\ (\mathcal{F}^e_{\pm}(z))^2 - \int_{J_e} \dfrac{(16\mathrm{i}\psi^e_V(\xi)(\mathcal{H}\psi^e_V)(\xi)-\frac{8\mathrm{i}\psi^e_V(\xi)}{\pi\xi})}{(\xi-z)}\dfrac{\mathrm{d}\xi}{2\pi\mathrm{i}}, & z\notin J_e, \end{cases}$$

where $\star^e_{\pm}(z):=\lim_{\varepsilon\downarrow 0}\star^e(z\pm \mathrm{i}0)$, $\star\in\{\mathcal{H},\mathcal{F}\}$. Recall the definition of $g^e(z)$ given in Lemma 3.4:

$$g^e(z):=\int_{J_e}\ln\left(\frac{(z-s)^2}{zs}\right)\mathrm{d}\mu^e_V(s)=\int_{J_e}\ln\left(\frac{(z-s)^2}{zs}\right)\psi^e_V(s)\,\mathrm{d}s, \quad z\in\mathbb{C}\setminus(-\infty,\max\{0,\max\{J_e\}\});$$

using the above distributional identities and the fact that $\int_{J_e}\psi^e_V(s)\,\mathrm{d}s=1$, one shows that

$$(g^e_{\pm}(z))' := \lim_{\varepsilon\downarrow 0}(g^e)'(z\pm \mathrm{i}\varepsilon)= \begin{cases} -\frac{1}{z}-2⨍_{J_e}\frac{\psi^e_V(s)}{s-z}\,\mathrm{d}s \mp 2\pi\mathrm{i}\psi^e_V(z), & z\in J_e, \\ -\frac{1}{z}-2\int_{J_e}\frac{\psi^e_V(s)}{s-z}\,\mathrm{d}s, & z\notin J_e, \end{cases}$$

whence one concludes that

$$(g^e_+ + g^e_-)'(z) = -\frac{2}{z}-4⨍_{J_e}\frac{\psi^e_V(s)}{s-z}\,\mathrm{d}s = -\frac{2}{z}+4\pi(\mathcal{H}\psi^e_V)(z), \quad z\in J_e,$$

$$(g^e_+ - g^e_-)'(z) = \begin{cases} -4\pi\mathrm{i}\psi^e_V(z), & z\in J_e, \\ 0, & z\notin J_e. \end{cases}$$

Demanding that (see Lemma 3.6 below) $(g^e_+ + g^e_-)'(z)=\widetilde{V}'(z)$, $z\in J_e$, one shows from the above that, for $J_e\ni z$, $((g^e(z))'+\frac{1}{z})_+ + ((g^e(z))'+\frac{1}{z})_- = 4\pi(\mathcal{H}\psi^e_V)(z)=\frac{2}{z}+\widetilde{V}'(z) \Rightarrow$

$$(\mathcal{H}\psi^e_V)(z)=\frac{1}{2\pi z}+\frac{\widetilde{V}'(z)}{4\pi}, \quad z\in J_e. \tag{3.3}$$

From Equation (3.2) and the above distributional identities, one shows that

$$\mathcal{F}^e_{\pm}(z):=\lim_{\varepsilon\downarrow 0}\mathcal{F}^e(z\pm\mathrm{i}\varepsilon)= \begin{cases} -\frac{1}{\pi\mathrm{i}z}-2\mathrm{i}(\mathcal{H}\psi^e_V)(z)\mp 2\psi^e_V(z), & z\in J_e, \\ -\frac{1}{\pi\mathrm{i}}\left(\frac{1}{z}+2\int_{J_e}\frac{\psi^e_V(s)}{s-z}\,\mathrm{d}s\right), & z\notin J_e; \end{cases} \tag{3.4}$$

thus, for $z\in\mathbb{R}\setminus(J_e\cup\{0\})$, $\mathcal{F}^e_+(z)=\mathcal{F}^e_-(z)=-\frac{1}{\pi\mathrm{i}}(\frac{1}{z}+2\int_{J_e}\frac{\psi^e_V(s)}{s-z}\,\mathrm{d}s)$. Hence, for $z\notin J_e\cup\{0\}$, one deduces that $\mathcal{H}^e_+(z)=\mathcal{H}^e_-(z)$. For $z\in J_e$, one notes that

$$\mathcal{H}^e_+(z)-\mathcal{H}^e_-(z)=(\mathcal{F}^e_+(z))^2-(\mathcal{F}^e_-(z))^2-16\mathrm{i}\psi^e_V(z)(\mathcal{H}\psi^e_V)(z)+\frac{8\mathrm{i}\psi^e_V(z)}{\pi z},$$

and

$$(\mathcal{F}^e_{\pm}(z))^2=-\frac{1}{\pi^2z^2}+\frac{4(\mathcal{H}\psi^e_V)(z)}{\pi z}\mp\frac{4\mathrm{i}\psi^e_V(z)}{\pi z}-4((\mathcal{H}\psi^e_V)(z))^2\pm 8\mathrm{i}\psi^e_V(z)(\mathcal{H}\psi^e_V)(z)+(2\psi^e_V(z))^2,$$

whence $(\mathcal{F}^e_+(z))^2-(\mathcal{F}^e_-(z))^2=-\frac{8\mathrm{i}\psi^e_V(z)}{\pi z}+16\mathrm{i}\psi^e_V(z)(\mathcal{H}\psi^e_V)(z)\Rightarrow\mathcal{H}^e_+(z)-\mathcal{H}^e_-(z)=0$; thus, for $z\in J_e$, $\mathcal{H}^e_+(z)=\mathcal{H}^e_-(z)$. The above argument shows, therefore, that $\mathcal{H}^e(z)$ is analytic across $\mathbb{R}\setminus\{0\}$; in fact, $\mathcal{H}^e(z)$ is entire for $z\in\mathbb{C}^*$. Recalling that $\mu^e_V\in\mathcal{M}_1(\mathbb{R})$, in particular, $\int_{J_e}s^{-m}\,\mathrm{d}\mu^e_V(s)=\int_{J_e}s^{-m}\psi^e_V(s)\,\mathrm{d}s<\infty$, $m\in\mathbb{N}$, one shows that, for $|z/s|\ll 1$, with $s\in J_e$ and $z\notin J_e$, via the expansion $\frac{1}{z-s}=-\sum_{k=0}^{l}\frac{z^k}{s^{k+1}}+\frac{z^{l+1}}{s^{l+1}(z-s)}$, $l\in\mathbb{Z}^+_0$,

$$(\mathcal{F}^e(z))^2 \underset{z\to 0}{=} -\frac{1}{\pi^2z^2}-\frac{1}{z}\left(\frac{4}{\pi^2}\int_{J_e}s^{-1}\,\mathrm{d}\mu^e_V(s)\right)+\mathcal{O}(1),$$

whence, upon recalling the definition of $\mathcal{H}^e(z)$, in particular, for $|z/\xi| \ll 1$, with $\xi \in J_e$ and $z \notin J_e$, via the expansion $\frac{1}{z-\xi} = -\sum_{k=0}^{l} \frac{z^k}{\xi^{k+1}} + \frac{z^{l+1}}{\xi^{l+1}(z-\xi)}$, $l \in \mathbb{Z}_0^+$,

$$\int_{J_e} \frac{(16\mathrm{i}\psi_V^e(\xi)(\mathcal{H}\psi_V^e)(\xi) - \frac{8\mathrm{i}\psi_V^e(\xi)}{\pi\xi})}{(\xi - z)} \frac{\mathrm{d}\xi}{2\pi\mathrm{i}} \underset{z\to 0}{=} \mathcal{O}(1),$$

it follows that

$$\mathcal{H}^e(z) \underset{z\to 0}{=} -\frac{1}{\pi^2 z^2} - \frac{1}{z}\left(\frac{4}{\pi^2}\int_{J_e} s^{-1}\,\mathrm{d}\mu_V^e(s)\right) + \mathcal{O}(1),$$

which shows that $\mathcal{H}^e(z)$ has a pole of order 2 at $z=0$, with $\mathrm{Res}(\mathcal{H}^e(z);0) = -4\pi^{-2}\int_{J_e} s^{-1}\,\mathrm{d}\mu_V^e(s)$. One learns from the above analysis that $z^2\mathcal{H}^e(z)$ is entire: look, in particular, at the behaviour of $z^2\mathcal{H}^e(z)$ as $|z| \to \infty$. Recalling Equations (3.1) and (3.2), one shows that, for $\mu_V^e \in \mathcal{M}_1(\mathbb{R})$, in particular, $\int_{J_e} \mathrm{d}\mu_V^e(s) = 1$ and $\int_{J_e} s^m\,\mathrm{d}\mu_V^e(s) < \infty$, $m \in \mathbb{N}$, for $|s/z| \ll 1$, with $s \in J_e$ and $z \notin J_e$, via the expansion $\frac{1}{s-z} = -\sum_{k=0}^{l} \frac{s^k}{z^{k+1}} + \frac{s^{l+1}}{z^{l+1}(s-z)}$, $l \in \mathbb{Z}_0^+$,

$$\begin{aligned} z^2\mathcal{H}^e(z) + \frac{1}{\pi^2} - \int_{J_e} s\left(16\mathrm{i}\psi_V^e(s)(\mathcal{H}\psi_V^e)(s) - \frac{8\mathrm{i}\psi_V^e(s)}{\pi s}\right)\frac{\mathrm{d}s}{2\pi\mathrm{i}} \\ - z\int_{J_e}\left(16\mathrm{i}\psi_V^e(s)(\mathcal{H}\psi_V^e)(s) - \frac{8\mathrm{i}\psi_V^e(s)}{\pi s}\right)\frac{\mathrm{d}s}{2\pi\mathrm{i}} \underset{|z|\to\infty}{=} \mathcal{O}(z^{-1}); \end{aligned}$$

thus, due to the entirety of $\mathcal{H}^e(z)$, it follows, by a generalisation of Liouville's Theorem, that

$$\begin{aligned} z^2\mathcal{H}^e(z) + \frac{1}{\pi^2} - \int_{J_e} s\left(16\mathrm{i}\psi_V^e(s)(\mathcal{H}\psi_V^e)(s) - \frac{8\mathrm{i}\psi_V^e(s)}{\pi s}\right)\frac{\mathrm{d}s}{2\pi\mathrm{i}} \\ - z\int_{J_e}\left(16\mathrm{i}\psi_V^e(s)(\mathcal{H}\psi_V^e)(s) - \frac{8\mathrm{i}\psi_V^e(s)}{\pi s}\right)\frac{\mathrm{d}s}{2\pi\mathrm{i}} = 0. \end{aligned}$$

Substituting Equation (3.1) into the above formula, one notes that

$$\begin{aligned} (\mathcal{F}^e(z))^2 - \int_{J_e} \frac{(16\mathrm{i}\psi_V^e(\xi)(\mathcal{H}\psi_V^e)(\xi) - \frac{8\mathrm{i}\psi_V^e(\xi)}{\pi\xi})}{(\xi - z)}\frac{\mathrm{d}\xi}{2\pi\mathrm{i}} - \frac{1}{z}\int_{J_e}\left(16\mathrm{i}\psi_V^e(\xi)(\mathcal{H}\psi_V^e)(\xi) - \frac{8\mathrm{i}\psi_V^e(\xi)}{\pi\xi}\right)\frac{\mathrm{d}\xi}{2\pi\mathrm{i}} \\ + \frac{1}{\pi^2 z^2} - \frac{1}{z^2}\int_{J_e}\xi\left(16\mathrm{i}\psi_V^e(\xi)(\mathcal{H}\psi_V^e)(\xi) - \frac{8\mathrm{i}\psi_V^e(\xi)}{\pi\xi}\right)\frac{\mathrm{d}\xi}{2\pi\mathrm{i}} = 0. \end{aligned}$$

Via Equation (3.3), it follows that $16\mathrm{i}\psi_V^e(s)(\mathcal{H}\psi_V^e)(s) - \frac{8\mathrm{i}\psi_V^e(s)}{\pi s} = \frac{4\mathrm{i}\psi_V^e(s)\widetilde{V}'(s)}{\pi}$; substituting the latter expression into the above equation, and re-arranging, one obtains,

$$(\mathcal{F}^e(z))^2 - \frac{2}{\pi^2}\int_{J_e}\frac{\widetilde{V}'(\xi)\psi_V^e(\xi)}{\xi - z}\,\mathrm{d}\xi + \frac{1}{\pi^2 z^2} - \frac{2}{\pi^2 z^2}\int_{J_e}\xi\widetilde{V}'(\xi)\psi_V^e(\xi)\,\mathrm{d}\xi - \frac{2}{\pi^2 z}\int_{J_e}\widetilde{V}'(\xi)\psi_V^e(\xi)\,\mathrm{d}\xi = 0. \tag{3.5}$$

But

$$\begin{aligned} \frac{2}{\pi^2}\int_{J_e}\frac{\widetilde{V}'(\xi)\psi_V^e(\xi)}{\xi - z}\,\mathrm{d}\xi &= \frac{2}{\pi^2}\int_{J_e}\frac{(\widetilde{V}'(\xi) - \widetilde{V}'(z))\psi_V^e(\xi)}{\xi - z}\,\mathrm{d}\xi + \frac{2}{\pi^2}\int_{J_e}\frac{\widetilde{V}'(z)\psi_V^e(\xi)}{\xi - z}\,\mathrm{d}\xi \\ &= \frac{2}{\pi^2}\int_{J_e}\frac{(\widetilde{V}'(\xi) - \widetilde{V}'(z))\psi_V^e(\xi)}{\xi - z}\,\mathrm{d}\xi + \frac{\mathrm{i}\widetilde{V}'(z)}{\pi}\underbrace{\left(\frac{2}{\pi\mathrm{i}}\int_{J_e}\frac{\psi_V^e(\xi)}{\xi - z}\,\mathrm{d}\xi\right)}_{= -\mathcal{F}^e(z) - (\mathrm{i}\pi z)^{-1}} \\ &= \frac{2}{\pi^2}\int_{J_e}\frac{(\widetilde{V}'(\xi) - \widetilde{V}'(z))\psi_V^e(\xi)}{\xi - z}\,\mathrm{d}\xi - \frac{\mathrm{i}\widetilde{V}'(z)\mathcal{F}^e(z)}{\pi} - \frac{\widetilde{V}'(z)}{\pi^2 z}: \end{aligned}$$

substituting the above into Equation (3.5), one arrives at, upon completing the square and re-arranging terms,

$$\left(\mathcal{F}^{e}(z)+\frac{\mathrm{i}\widetilde{V}'(z)}{2\pi}\right)^{2}+\frac{\mathfrak{q}^{e}_{V}(z)}{\pi^{2}}=0, \tag{3.6}$$

where

$$\mathfrak{q}^{e}_{V}(z):=\left(\frac{\widetilde{V}'(z)}{2}\right)^{2}+\frac{\widetilde{V}'(z)}{z}-2\int_{J_{e}}\frac{(\widetilde{V}'(\xi)-\widetilde{V}'(z))\psi^{e}_{V}(\xi)}{\xi-z}\,\mathrm{d}\xi+\frac{1}{z^{2}}\left(1-2\int_{J_{e}}(\xi+z)\widetilde{V}'(\xi)\psi^{e}_{V}(\xi)\,\mathrm{d}\xi\right).$$

(Equation (3.6) above generalizes Equation (3.5) for $q^{(0)}(x)$ in [58] for the case when $\widetilde{V}\colon \mathbb{R}\setminus\{0\}\to\mathbb{R}$ is real analytic; moreover, it is analogous to Equation (1.37) of [56].) Note that, since $\widetilde{V}\colon \mathbb{R}\setminus\{0\}\to\mathbb{R}$ satisfies conditions (2.3)–(2.5), it follows from $\alpha^{l}-\beta^{l}=(\alpha-\beta)(\alpha^{l-1}+\alpha^{l-2}\beta+\cdots+\alpha\beta^{l-2}+\beta^{l-1})$, $l\in\mathbb{N}$, that $\mathfrak{q}^{e}_{V}(z)$ is real analytic on J_{e} (and real analytic on $\mathbb{R}\setminus\{0\}$). For $x\in J_{e}$, set $z:=x+\mathrm{i}\varepsilon$, and consider the $\varepsilon\downarrow 0$ limit of Equation (3.6): $\lim_{\varepsilon\downarrow 0}(\mathcal{F}^{e}(x+\mathrm{i}\varepsilon)+\frac{\mathrm{i}\widetilde{V}'(x+\mathrm{i}\varepsilon)}{2\pi})^{2}=(\mathcal{F}^{e}_{+}(x)+\frac{\mathrm{i}\widetilde{V}'(x)}{2\pi})^{2}$ (as $\widetilde{V}$ is real analytic on J_{e}); recalling that $\mathcal{F}^{e}_{+}(x)=-\frac{1}{\pi\mathrm{i}x}-2\mathrm{i}(\mathcal{H}\psi^{e}_{V})(x)-2\psi^{e}_{V}(x)$, via Equation (3.3), it follows that $\mathcal{F}^{e}_{+}(x)=-\frac{\mathrm{i}\widetilde{V}'(x)}{2\pi}-2\psi^{e}_{V}(x)\Rightarrow$ $(\mathcal{F}^{e}_{+}(x)+\frac{\mathrm{i}\widetilde{V}'(x)}{2\pi})^{2}=(2\psi^{e}_{V}(x))^{2}$, whence $(\psi^{e}_{V}(x))^{2}=-\mathfrak{q}^{e}_{V}(x)/(2\pi)^{2}$, $x\in J_{e}$, whereupon, using the fact that (see above) $\psi^{e}_{V}(x)\geqslant 0$ $\forall$ $x\in\overline{J_{e}}$, it follows that $\mathfrak{q}^{e}_{V}(x)\leqslant 0$, $x\in J_{e}$; moreover, as a by-product, decomposing $\mathfrak{q}^{e}_{V}(x)$, for $x\in J_{e}$, into positive and negative parts, that is, $\mathfrak{q}^{e}_{V}(x)=(\mathfrak{q}^{e}_{V}(x))^{+}-(\mathfrak{q}^{e}_{V}(x))^{-}$, $x\in J_{e}$, where $(\mathfrak{q}^{e}_{V}(x))^{\pm}:=\max\left\{\pm\mathfrak{q}^{e}_{V}(x),0\right\}$ $(\geqslant 0)$, one learns from the above analysis that, for $x\in J_{e}$, $(\mathfrak{q}^{e}_{V}(x))^{+}\equiv 0$ and $\psi^{e}_{V}(x)=\frac{1}{2\pi}((\mathfrak{q}^{e}_{V}(x))^{-})^{1/2}$; and, since $\int_{J_{e}}\psi^{e}_{V}(s)\,\mathrm{d}s=1$, it follows that $\frac{1}{2\pi}\int_{J_{e}}((\mathfrak{q}^{e}_{V}(s))^{-})^{1/2}\,\mathrm{d}s=1$, which gives rise to the interesting fact that the function $(\mathfrak{q}^{e}_{V}(x))^{-}\not\equiv 0$ on J_{e}. (Even though $(\mathfrak{q}^{e}_{V}(x))^{-}$ depends on $\mathrm{d}\mu^{e}_{V}(x)=\psi^{e}_{V}(x)\,\mathrm{d}x$, and thus $\psi^{e}_{V}(x)=\frac{1}{2\pi}((\mathfrak{q}^{e}_{V}(x))^{-})^{1/2}$ is an implicit representation for ψ^{e}_{V}, it is still a useful relation which can be used to obtain additional, valuable information about ψ^{e}_{V}.) For $x\notin J_{e}$, set $z:=x+\mathrm{i}\varepsilon$, and (again) study the $\varepsilon\downarrow 0$ limit of Equation (3.6): in this case, $\lim_{\varepsilon\downarrow 0}(\mathcal{F}^{e}(x+\mathrm{i}\varepsilon)+\frac{\mathrm{i}\widetilde{V}'(x+\mathrm{i}\varepsilon)}{2\pi})^{2}=$ $(\mathcal{F}^{e}_{+}(x)+\frac{\mathrm{i}\widetilde{V}'(x)}{2\pi})^{2}=(\mathcal{F}^{e}(x)+\frac{\mathrm{i}\widetilde{V}'(x)}{2\pi})^{2}$; recalling that, for $x\notin J_{e}$, $\mathcal{F}^{e}(x)=-\frac{1}{\pi\mathrm{i}}(\frac{1}{x}+2\int_{J_{e}}\frac{\psi^{e}_{V}(s)}{s-x}\,\mathrm{d}s)=\frac{\mathrm{i}}{\pi x}-2\mathrm{i}(\mathcal{H}\psi^{e}_{V})(x)$, substituting the latter expression into Equation (3.6), one arrives at $(\frac{1}{\pi x}-2(\mathcal{H}\psi^{e}_{V})(x)+\frac{\widetilde{V}'(x)}{2\pi})^{2}=\mathfrak{q}^{e}_{V}(x)/\pi^{2}$, $x\notin J_{e}$ (since $\widetilde{V}'$ is real analytic on $(\mathbb{R}\setminus\{0\})\setminus J_{e}$, it follows that $\mathfrak{q}^{e}_{V}(x)$, too, is real analytic on $(\mathbb{R}\setminus\{0\})\setminus J_{e}$, in which case, this latter relation merely states that, for $x=0$, $+\infty=+\infty$), whence $\mathfrak{q}^{e}_{V}(x)\geqslant 0$ $\forall$ $x\notin J_{e}$.

Now, recalling that, on a compact subset of $\mathbb{R}$, an analytic function changes sign an at most countable number of times, it follows from the above argument, the fact that $\widetilde{V}\colon\mathbb{R}\setminus\{0\}\to\mathbb{R}$ satisfying conditions (2.3)–(2.5) is regular (cf. Subsection 2.2), in particular, $\widetilde{V}$ is real analytic in the (open) neighbourhood $\mathbb{U}:=\{z\in\mathbb{C};\ \inf_{q\in J_{e}}|z-q|<r\in(0,1)\}\setminus\{0\}$, μ^{e}_{V} has compact support, and mimicking a part of the calculations subsumed in the proof of Theorem 1.38 in [56], that $J_{e}:=\operatorname{supp}(\mu^{e}_{V})=\{x\in\mathbb{R};\ \mathfrak{q}^{e}_{V}(x)\leqslant 0\}$ consists of the disjoint union of a finite number of bounded (real) intervals, with representation $J_{e}:=\cup_{j=1}^{N+1}J^{e}_{j}$, where $J^{e}_{j}:=[b^{e}_{j-1},a^{e}_{j}]$, with $N\in\mathbb{N}$ and finite, $b^{e}_{0}:=\min\{J_{e}\}\notin\{-\infty,0\}$, $a^{e}_{N+1}:=\max\{J_{e}\}\notin\{0,+\infty\}$, and $-\infty<b^{e}_{0}<a^{e}_{1}<b^{e}_{1}<a^{e}_{2}<\cdots<b^{e}_{N}<a^{e}_{N+1}<+\infty$. (One notes that $\widetilde{V}$ is real analytic in, say, the open neighbourhood $\widetilde{\mathbb{U}}:=\cup_{j=1}^{N+1}\widetilde{\mathbb{U}}_{j}$, where $\widetilde{\mathbb{U}}_{j}:=\{z\in\mathbb{C}^{*};\ \inf_{q\in J^{e}_{j}}|z-q|<r_{j}\in(0,1)\}$, with $\widetilde{\mathbb{U}}_{i}\cap\widetilde{\mathbb{U}}_{j}=\varnothing$, $i\neq j=1,\dots,N+1$.) Furthermore, as a by-product of the above representation for J_{e}, it follows that, since $J^{e}_{i}\cap J^{e}_{j}=\varnothing$, $i\neq j=1,\dots,N+1$, $\operatorname{meas}(J_{e})=\sum_{j=1}^{N+1}|b^{e}_{j-1}-a^{e}_{j}|<+\infty$.

It remains, still, to determine the $2(N+1)$ conditions satisfied by the end-points of the support of the 'even' equilibrium measure, $\{b^{e}_{j-1},a^{e}_{j}\}_{j=1}^{N+1}$. Towards this end, one proceeds as follows. From the formula for $\mathcal{F}^{e}(z)$ given in Equation (3.2):

(i) for $\mu^{e}_{V}\in\mathcal{M}_{1}(\mathbb{R})$, in particular, $\int_{\mathbb{R}}\mathrm{d}\mu^{e}_{V}(s)=1$ and $\int_{\mathbb{R}}s^{m}\,\mathrm{d}\mu^{e}_{V}(s)<\infty$, $m\in\mathbb{N}$, $s\in J_{e}$ and $z\notin J_{e}$, with $|s/z|\ll 1$ (e.g., $|z|\gg\max_{j=1,\dots,N+1}\{|b^{e}_{j-1}-a^{e}_{j}|\}$), via the expansion $\frac{1}{s-z}=-\sum_{k=0}^{l}\frac{s^{k}}{z^{k+1}}+\frac{s^{l+1}}{z^{l+1}(s-z)}$, $l\in\mathbb{Z}^{+}_{0}$, one gets that $\mathcal{F}^{e}(z)=_{z\to\infty}\frac{1}{\pi\mathrm{i}z}+\mathcal{O}(z^{-2})$;

(ii) for $\mu^{e}_{V}\in\mathcal{M}_{1}(\mathbb{R})$, in particular, $\int_{\mathbb{R}}s^{-m}\,\mathrm{d}\mu^{e}_{V}(s)<\infty$, $m\in\mathbb{N}$, $s\in J_{e}$ and $z\notin J_{e}$, with $|z/s|\ll 1$ (e.g., $|z|\ll\min_{j=1,\dots,N+1}\{|b^{e}_{j-1}-a^{e}_{j}|\}$), via the expansion $\frac{1}{z-s}=-\sum_{k=0}^{l}\frac{z^{k}}{s^{k+1}}+\frac{z^{l+1}}{s^{l+1}(z-s)}$, $l\in\mathbb{Z}^{+}_{0}$, one gets that $\mathcal{F}^{e}(z)=_{z\to 0}-\frac{1}{\pi\mathrm{i}z}+\mathcal{O}(1)$.

Recalling, also, the formulae for $\mathcal{F}^{e}_{\pm}(z)$ given in Equation (3.4), one deduces that $\mathcal{F}^{e}_{+}(z)+\mathcal{F}^{e}_{-}(z)=-\mathrm{i}\widetilde{V}'(z)/\pi$,

$z \in J_e$, and $\mathcal{F}^e_+(z) - \mathcal{F}^e_-(z) = 0$, $z \notin J_e$; thus, one learns that $\mathcal{F}^e \colon \mathbb{C} \setminus (J_e \cup \{0\}) \to \mathbb{C}$ solves the following (scalar and homogeneous) RHP:

(1) $\mathcal{F}^e(z)$ is holomorphic (resp., meromorphic) for $z \in \mathbb{C} \setminus (J_e \cup \{0\})$ (resp., $z \in \mathbb{C} \setminus J_e$);

(2) $\mathcal{F}^e_\pm(z) := \lim_{\varepsilon \downarrow 0} \mathcal{F}^e(z \pm \mathrm{i}\varepsilon)$ satisfy the boundary condition $\mathcal{F}^e_+(z) + \mathcal{F}^e_-(z) = -\mathrm{i}\widetilde{V}'(z)/\pi$, $z \in J_e$, with $\mathcal{F}^e_+(z) = \mathcal{F}^e_-(z) := \mathcal{F}^e(z)$ for $z \notin J_e$;

(3) $\mathcal{F}^e(z) =_{\substack{z\to\infty \\ z \in \mathbb{C}\setminus\mathbb{R}}} \frac{1}{\pi \mathrm{i} z} + \mathcal{O}(z^{-2})$; and

(4) $\operatorname{Res}(\mathcal{F}^e(z); 0) = -1/\pi \mathrm{i}$.

The solution of this RHP is (see, for example, [95])

$$\mathcal{F}^e(z) = -\frac{1}{\pi \mathrm{i} z} + (R_e(z))^{1/2} \int_{J_e} \frac{(\frac{2}{\mathrm{i}\pi s} + \frac{\widetilde{V}'(s)}{\mathrm{i}\pi})}{(R_e(s))^{1/2}_+ (s-z)} \frac{\mathrm{d}s}{2\pi \mathrm{i}}, \quad z \in \mathbb{C} \setminus (J_e \cup \{0\}),$$

where $(R_e(z))^{1/2}$ is defined in the Lemma, with $(R_e(z))^{1/2}_\pm := \lim_{\varepsilon\downarrow 0}(R_e(z \pm \mathrm{i}\varepsilon))^{1/2}$, and the branch of the square root is chosen so that $z^{-(N+1)}(R_e(z))^{1/2} \sim_{\substack{z\to\infty \\ z\in\mathbb{C}_\pm}} \pm 1$. (Note that $(R_e(z))^{1/2}$ is pure imaginary on J_e.) It follows from the above integral representation for $\mathcal{F}^e(z)$ that, for $s \in J_e$ and $z \notin J_e$, with $|s/z| \ll 1$ (e.g., $|z| \gg \max_{j=1,\dots,N+1}\{|b^e_{j-1} - a^e_j|\}$), via the expansion $\frac{1}{s-z} = -\sum_{k=0}^{l} \frac{s^k}{z^{k+1}} + \frac{s^{l+1}}{z^{l+1}(s-z)}$, $l \in \mathbb{Z}^+_0$,

$$\mathcal{F}^e(z) \underset{z\to\infty}{=} -\frac{1}{\mathrm{i}\pi z} + \frac{(z^{N+1} + \cdots)}{z} \int_{J_e} \frac{(\frac{2\mathrm{i}}{\pi s} + \frac{\mathrm{i}\widetilde{V}'(s)}{\pi})}{(R_e(s))^{1/2}_+} \left(1 + \frac{s}{z} + \cdots + \frac{s^N}{z^N} + \frac{s^{N+1}}{z^{N+1}} + \cdots\right) \frac{\mathrm{d}s}{2\pi \mathrm{i}} :$$

now, recalling from above that $\mathcal{F}^e(z) =_{z\to\infty} \frac{1}{\pi \mathrm{i} z} + \mathcal{O}(z^{-2})$, it follows that, upon removing the secular (growing) terms,

$$\int_{J_e} \left(\frac{2\mathrm{i}}{\pi s} + \frac{\mathrm{i}\widetilde{V}'(s)}{\pi}\right) \frac{s^j}{(R_e(s))^{1/2}_+} \frac{\mathrm{d}s}{2\pi \mathrm{i}} = 0, \quad j = 0, \dots, N$$

(which gives $N+1$ (real) moment conditions), and, upon equating z^{-1} terms,

$$\int_{J_e} \left(\frac{2\mathrm{i}}{\pi s} + \frac{\mathrm{i}\widetilde{V}'(s)}{\pi}\right) \frac{s^{N+1}}{(R_e(s))^{1/2}_+} \frac{\mathrm{d}s}{2\pi \mathrm{i}} = \frac{2}{\pi \mathrm{i}};$$

it remains, therefore, to determine an additional $2(N+1) - (N+1) - 1 = N$ (real) moment conditions. From the integral representation for $\mathcal{F}^e(z)$, a residue calculus calculation shows that

$$\mathcal{F}^e(z) = -\frac{\mathrm{i}\widetilde{V}'(z)}{2\pi} - \frac{(R_e(z))^{1/2}}{2} \oint_{C^e_{\mathrm{R}}} \frac{(\frac{2\mathrm{i}}{\pi s} + \frac{\mathrm{i}\widetilde{V}'(s)}{\pi})}{(R_e(s))^{1/2}(s-z)} \frac{\mathrm{d}s}{2\pi \mathrm{i}}, \tag{3.7}$$

where C^e_{R} ($\subset \mathbb{C}^*$) denotes the boundary of any open doubly-connected annular region of the type $\{z' \in \mathbb{C};\ 0 < r < |z'| < R < +\infty\}$, where the simple outer (resp., inner) boundary $\{z' = R\mathrm{e}^{\mathrm{i}\vartheta},\ 0 \leqslant \vartheta \leqslant 2\pi\}$ (resp., $\{z' = r\mathrm{e}^{\mathrm{i}\vartheta},\ 0 \leqslant \vartheta \leqslant 2\pi\}$) is traversed clockwise (resp., counter-clockwise), with the numbers $0 < r < R < +\infty$ chosen such that, for (any) non-real z in the domain of analyticity of $\widetilde{V}$ (that is, $\mathbb{C}^*$), $\operatorname{int}(C^e_{\mathrm{R}}) \supset J_e \cup \{z\}$. Recall from Equation (3.4) that, for $z \in \mathbb{R} \setminus \overline{J_e}$ ($\supset \cup_{j=1}^N (a^e_j, b^e_j)$), $\mathcal{F}^e_+(z) = \mathcal{F}^e_-(z) = -\frac{1}{\pi \mathrm{i}}(\frac{1}{z} + 2\int_{J_e} \frac{\psi^e_V(s)}{s-z}\,\mathrm{d}s)$, whence $\mathcal{F}^e(z) + \frac{1}{\pi \mathrm{i} z} = -2\mathrm{i}(\mathcal{H}\psi^e_V)(z)$; thus, using Equation (3.7), one arrives at

$$(\mathcal{H}\psi^e_V)(z) = \frac{\widetilde{V}'(z)}{4\pi} + \frac{1}{2\pi z} + \frac{\mathrm{i}(R_e(z))^{1/2}}{2} \oint_{C^e_{\mathrm{R}}} \frac{(\frac{1}{\pi \mathrm{i}\xi} + \frac{\widetilde{V}'(\xi)}{2\pi \mathrm{i}})}{(R_e(\xi))^{1/2}(\xi - z)} \frac{\mathrm{d}\xi}{2\pi \mathrm{i}}, \quad z \in \cup_{j=1}^N (a^e_j, b^e_j).$$

A contour integration argument shows that

$$\int_{a^e_j}^{b^e_j} \left((\mathcal{H}\psi^e_V)(s) - \frac{1}{2\pi s} - \frac{\widetilde{V}'(s)}{4\pi}\right) \mathrm{d}s = 0, \quad j = 1, \dots, N,$$

whence, using the above expression for $(\mathcal{H}\psi^e_V)(z)$, $z \in \cup_{j=1}^N (a^e_j, b^e_j)$, it follows that

$$\int_{a^e_j}^{b^e_j} \left(\frac{\mathrm{i}(R_e(s))^{1/2}}{2} \oint_{C^e_{\mathrm{R}}} \frac{(\frac{1}{\pi \mathrm{i}\xi} + \frac{\widetilde{V}'(\xi)}{2\pi \mathrm{i}})}{(R_e(\xi))^{1/2}(\xi - s)} \frac{\mathrm{d}\xi}{2\pi \mathrm{i}}\right) \mathrm{d}s = 0, \quad j = 1, \dots, N : \tag{3.8}$$

now, 'collapsing' the contour C^e_{R} down to $\mathbb{R}\setminus\{0\}$ and using the Residue Theorem, one shows that

$$\frac{\mathrm{i}(R_e(z))^{1/2}}{2}\oint_{C^e_{\mathrm{R}}}\frac{(\frac{1}{\pi\mathrm{i}\xi}+\frac{\widetilde{V}'(\xi)}{2\pi\mathrm{i}})}{(R_e(\xi))^{1/2}(\xi-z)}\frac{\mathrm{d}\xi}{2\pi\mathrm{i}}=-\frac{1}{2\pi z}-\frac{\widetilde{V}'(z)}{4\pi}+\mathrm{i}(R_e(z))^{1/2}\int_{J_e}\frac{(\frac{1}{\pi\mathrm{i}\xi}+\frac{\widetilde{V}'(\xi)}{2\pi\mathrm{i}})}{(R_e(\xi))^{1/2}_+(\xi-z)}\frac{\mathrm{d}\xi}{2\pi\mathrm{i}};$$

substituting the latter relation into Equation (3.8), one arrives at, after straightforward integration and using the Fundamental Theorem of Calculus, for $j=1,\dots,N$,

$$\int_{a^e_j}^{b^e_j}\left(\mathrm{i}(R_e(s))^{1/2}\int_{J_e}\left(\frac{\mathrm{i}}{\pi\xi}+\frac{\mathrm{i}\widetilde{V}'(\xi)}{2\pi}\right)\frac{1}{(R_e(\xi))^{1/2}_+(\xi-s)}\frac{\mathrm{d}\xi}{2\pi\mathrm{i}}\right)\mathrm{d}s=\frac{1}{2\pi}\ln\left|\frac{a^e_j}{b^e_j}\right|+\frac{1}{4\pi}\left(\widetilde{V}(a^e_j)-\widetilde{V}(b^e_j)\right),$$

which give the remaining N moment conditions determining the end-points of the support of the 'even' equilibrium measure, $\{b^e_{j-1},a^e_j\}_{j=1}^{N+1}$. Since $J_e\not\supseteq\{0,\pm\infty\}$ and $\widetilde{V}$ is real analytic on J_e,

$$(R_e(s))^{1/2}\underset{s\downarrow b^e_{j-1}}{=}\mathcal{O}\left((s-b^e_{j-1})^{1/2}\right)\qquad\text{and}\qquad(R_e(s))^{1/2}\underset{s\uparrow a^e_j}{=}\mathcal{O}\left((a^e_j-s)^{1/2}\right),\quad j=1,\dots,N+1,$$

which shows that all the integrals above constituting the n-dependent system of $2(N+1)$ moment conditions for the end-points of the support of μ^e_V have removable singularities at b^e_{j-1},a^e_j, $j=1,\dots,N+1$.

Recall from Equation (3.4) that, for $z\in J_e$, $\mathcal{F}^e_{\pm}(z)=-\frac{1}{\pi\mathrm{i}z}-2\mathrm{i}(\mathcal{H}\psi^e_V)(z)\mp2\psi^e_V(z)$: using the fact that, from Equation (3.3), for $z\in J_e$, $(\mathcal{H}\psi^e_V)(z)=\frac{1}{2\pi z}+\frac{\widetilde{V}'(z)}{4\pi}$, it follows that

$$\mathcal{F}^e_{\pm}(z)=\frac{\widetilde{V}'(z)}{2\pi\mathrm{i}}\mp2\psi^e_V(z),\quad z\in J_e.$$

From Equation (3.7), it follows that

$$\mathcal{F}^e_{\pm}(z)=\frac{\widetilde{V}'(z)}{2\pi\mathrm{i}}+\frac{(R_e(z))^{1/2}_{\pm}}{2}\oint_{C^e_{\mathrm{R}}}\frac{(\frac{2}{\pi\mathrm{i}s}+\frac{\widetilde{V}'(s)}{\mathrm{i}\pi})}{(R_e(s))^{1/2}(s-z)}\frac{\mathrm{d}s}{2\pi\mathrm{i}};$$

thus, equating the above two expressions for $\mathcal{F}^e_{\pm}(z)$, one arrives at $\psi^e_V(x)=\frac{1}{2\pi\mathrm{i}}(R_e(x))^{1/2}_+h^e_V(x)\mathbf{1}_{J_e}(x)$, where $h^e_V(z)$ is defined in the Lemma, and $\mathbf{1}_{J_e}(x)$ is the characteristic function of the set J_e, which gives rise to the formula for the density of the 'even' equilibrium measure, $\mathrm{d}\mu^e_V(x)=\psi^e_V(x)\,\mathrm{d}x$ (the integral representation for $h^e_V(z)$ shows that it is analytic in some open subset of $\mathbb{C}^*$ containing J_e). Now, recalling that $\widetilde{V}\colon\mathbb{R}\setminus\{0\}\to\mathbb{R}$ satisfying conditions (2.3)–(2.5) is regular, and that, for $s\in J_e$ (resp., $s\in\overline{J_e}$), $\psi^e_V(s)>0$ (resp., $\psi^e_V(s)\geqslant0$) and $(R_e(s))^{1/2}_+=\mathrm{i}(|R_e(s)|)^{1/2}\in\mathrm{i}\mathbb{R}_{\pm}$ (resp., $(R_e(s))^{1/2}_+=\mathrm{i}(|R_e(s)|)^{1/2}\in\mathrm{i}\mathbb{R}$), it follows from the formula $\psi^e_V(s)=\frac{1}{2\pi\mathrm{i}}(R_e(s))^{1/2}_+h^e_V(s)\mathbf{1}_{J_e}(s)$ and the regularity assumption, namely, $h^e_V(z)\not\equiv0$ for $z\in\overline{J_e}$, that $(|R_e(s)|)^{1/2}h^e_V(s)>0$, $s\in J_e$ (resp., $(|R_e(s)|)^{1/2}h^e_V(s)\geqslant0$, $s\in\overline{J_e}$).

Finally, it will be shown that, if $\overline{J_e}:=\cup_{j=1}^{N+1}[b^e_{j-1},a^e_j]$, the end-points of the support of the 'even' equilibrium measure, which satisfy the n-dependent system of $2(N+1)$ moment conditions stated in the Lemma, are (real) analytic functions of z_o, thus proving the (local) solvability of the n-dependent $2(N+1)$ moment conditions. Towards this end, one follows closely the idea of the proof of Theorem 1.3 (iii) in [92] (see, also, Section 8 of [56], and [96]). Recall from Subsection 2.2 that $\widetilde{V}(z):=z_oV(z)$, where $z_o\colon\mathbb{N}\times\mathbb{N}\to\mathbb{R}_+$, $(n,\mathcal{N})\mapsto z_o:=\mathcal{N}/n$, and, in the double-scaling limit as $\mathcal{N},n\to\infty$, $z_o=1+o(1)$. Furthermore, in the analysis above, it was shown that the end-points of the support of the 'even' equilibrium measure were the simple zeros/roots of the function $\mathfrak{q}^e_V(z)$, that is (up to re-arrangement), $\{b^e_0,b^e_1,\dots,b^e_N,a^e_1,a^e_2,\dots,a^e_{N+1}\}=\{x\in\mathbb{R};\ \mathfrak{q}^e_V(x)=0\}$ (these are the only roots for the regular case studied in this work). The function $\mathfrak{q}^e_V(x)\in\mathbb{R}(x)$ (the algebra of rational functions in x with coefficients in $\mathbb{R}$) is real rational on $\mathbb{R}$ and real analytic on $\mathbb{R}\setminus\{0\}$, it has analytic continuation to $\{z\in\mathbb{C};\ \inf_{p\in\mathbb{R}}|z-p|<r\in(0,1)\}\setminus\{0\}$ (independent of z_o), and depends continuously on z_o; thus, its simple zeros/roots, that is, $b^e_{k-1}=b^e_{k-1}(z_o)$ and $a^e_k=a^e_k(z_o)$, $k=1,\dots,N+1$, are continuous functions of z_o.

Write the large-z (e.g., $|z|\gg\max_{j=1,\dots,N+1}\{|b^e_{j-1}-a^e_j|\}$) asymptotic expansion for $\mathcal{F}^e(z)$ given above as follows:

$$\mathcal{F}^e(z)\underset{z\to\infty}{=}-\frac{1}{\mathrm{i}\pi z}-\frac{(R_e(z))^{1/2}}{2\pi\mathrm{i}z}\sum_{j=0}^{\infty}\mathcal{T}^e_jz^{-j},$$

where

$$\mathcal{T}_j^e := \int_{J_e} \left(\frac{2}{\mathrm{i}\pi s} + \frac{\widetilde{V}'(s)}{\mathrm{i}\pi} \right) \frac{s^j}{(R_e(s))_+^{1/2}} \, \mathrm{d}s, \quad j \in \mathbb{Z}_0^+.$$

Set

$$\mathcal{N}_j^e := \int_{a_j^e}^{b_j^e} \left((\mathcal{H}\psi_V^e)(s) - \frac{1}{2\pi s} - \frac{\widetilde{V}'(s)}{4\pi} \right) \mathrm{d}s, \quad j = 1, \dots, N.$$

The (n-dependent) $2(N+1)$ moment conditions are, thus, equivalent to $\mathcal{T}_j^e = 0$, $j = 0, \dots, N$, $\mathcal{T}_{N+1}^e = -4$, and $\mathcal{N}_j^e = 0$, $j = 1, \dots, N$. It will first be shown that, for regular $\widetilde{V} \colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ satisfying conditions (2.3)–(2.5), the Jacobian of the transformation $\{b_0^e(z_o), \dots, b_N^e(z_o), a_1^e(z_o), \dots, a_{N+1}^e(z_o)\} \mapsto \{\mathcal{T}_0^e, \dots, \mathcal{T}_{N+1}^e, \mathcal{N}_1^e, \dots, \mathcal{N}_N^e\}$, that is, $\mathrm{Jac}(\mathcal{T}_0^e, \dots, \mathcal{T}_{N+1}^e, \mathcal{N}_1^e, \dots, \mathcal{N}_N^e) := \frac{\partial(\mathcal{T}_0^e, \dots, \mathcal{T}_{N+1}^e, \mathcal{N}_1^e, \dots, \mathcal{N}_N^e)}{\partial(b_0^e, \dots, b_N^e, a_1^e, \dots, a_{N+1}^e)}$, is non-zero whenever $b_{j-1}^e = b_{j-1}^e(z_o)$ and $a_j^e = a_j^e(z_o)$, $j = 1, \dots, N+1$, are chosen so that $\overline{J_e} = \cup_{j=1}^{N+1} [b_{j-1}^e, a_j^e]$. Using the equation $(\mathcal{H}\psi_V^e)(z) = \frac{\mathrm{i}}{2} \mathfrak{F}^e(z) + \frac{1}{2\pi z}$ (cf. Equation (3.2)), one follows the analysis on pp. 778–779 of [92] to show that, for $k = 1, \dots, N+1$:

$$\frac{\partial \mathcal{T}_j^e}{\partial b_{k-1}^e} = b_{k-1}^e \frac{\partial \mathcal{T}_{j-1}^e}{\partial b_{k-1}^e} + \frac{1}{2} \mathcal{T}_{j-1}^e, \quad j \in \mathbb{N}, \tag{T1}$$

$$\frac{\partial \mathcal{T}_j^e}{\partial a_k^e} = a_k^e \frac{\partial \mathcal{T}_{j-1}^e}{\partial a_k^e} + \frac{1}{2} \mathcal{T}_{j-1}^e, \quad j \in \mathbb{N}, \tag{T2}$$

$$\frac{\partial \mathfrak{F}^e(z)}{\partial b_{k-1}^e} = -\frac{1}{2\pi \mathrm{i}} \left(\frac{\partial \mathcal{T}_0^e}{\partial b_{k-1}^e} \right) \frac{(R_e(z))^{1/2}}{z - b_{k-1}^e}, \quad z \in \mathbb{C} \setminus (\overline{J_e} \cup \{0\}), \tag{F1}$$

$$\frac{\partial \mathfrak{F}^e(z)}{\partial a_k^e} = -\frac{1}{2\pi \mathrm{i}} \left(\frac{\partial \mathcal{T}_0^e}{\partial a_k^e} \right) \frac{(R_e(z))^{1/2}}{z - a_k^e}, \quad z \in \mathbb{C} \setminus (\overline{J_e} \cup \{0\}), \tag{F2}$$

$$\frac{\partial \mathcal{N}_j^e}{\partial b_{k-1}^e} = -\frac{1}{4\pi} \left(\frac{\partial \mathcal{T}_0^e}{\partial b_{k-1}^e} \right) \int_{a_j^e}^{b_j^e} \frac{(R_e(s))^{1/2}}{s - b_{k-1}^e} \, \mathrm{d}s, \quad j = 1, \dots, N, \tag{N1}$$

$$\frac{\partial \mathcal{N}_j^e}{\partial a_k^e} = -\frac{1}{4\pi} \left(\frac{\partial \mathcal{T}_0^e}{\partial a_k^e} \right) \int_{a_j^e}^{b_j^e} \frac{(R_e(s))^{1/2}}{s - a_k^e} \, \mathrm{d}s, \quad j = 1, \dots, N; \tag{N2}$$

furthermore, if one evaluates Equations (T1) and (T2) on the solution of the n-dependent system of $2(N+1)$ moment conditions, that is, $\mathcal{T}_j^e = 0$, $j = 0, \dots, N$, $\mathcal{T}_{N+1}^e = -4$, and $\mathcal{N}_i^e = 0$, $i = 1, \dots, N$, one arrives at

$$\frac{\partial \mathcal{T}_j^e}{\partial b_{k-1}^e} = (b_{k-1}^e)^j \frac{\partial \mathcal{T}_0^e}{\partial b_{k-1}^e}, \qquad \frac{\partial \mathcal{T}_j^e}{\partial a_k^e} = (a_k^e)^j \frac{\partial \mathcal{T}_0^e}{\partial a_k^e}, \quad j = 0, \dots, N+1. \tag{S1}$$

Via Equations (N1), (N2), and (S1), one now computes the Jacobian of the transformation $\{b_0^e(z_o), \dots, b_N^e(z_o), a_1^e(z_o), \dots, a_{N+1}^e(z_o)\} \mapsto \{\mathcal{T}_0^e, \dots, \mathcal{T}_{N+1}^e, \mathcal{N}_1^e, \dots, \mathcal{N}_N^e\}$ on the solution of the n-dependent system of $2(N+1)$ moment conditions:

$$\mathrm{Jac}(\mathcal{T}_0^e, \dots, \mathcal{T}_{N+1}^e, \mathcal{N}_1^e, \dots, \mathcal{N}_N^e) := \frac{\partial(\mathcal{T}_0^e, \dots, \mathcal{T}_{N+1}^e, \mathcal{N}_1^e, \dots, \mathcal{N}_N^e)}{\partial(b_0^e, \dots, b_N^e, a_1^e, \dots, a_{N+1}^e)}$$

$$= \begin{vmatrix}
\frac{\partial \mathcal{T}_0^e}{\partial b_0^e} & \frac{\partial \mathcal{T}_0^e}{\partial b_1^e} & \cdots & \frac{\partial \mathcal{T}_0^e}{\partial b_N^e} & \frac{\partial \mathcal{T}_0^e}{\partial a_1^e} & \frac{\partial \mathcal{T}_0^e}{\partial a_2^e} & \cdots & \frac{\partial \mathcal{T}_0^e}{\partial a_{N+1}^e} \\
\frac{\partial \mathcal{T}_1^e}{\partial b_0^e} & \frac{\partial \mathcal{T}_1^e}{\partial b_1^e} & \cdots & \frac{\partial \mathcal{T}_1^e}{\partial b_N^e} & \frac{\partial \mathcal{T}_1^e}{\partial a_1^e} & \frac{\partial \mathcal{T}_1^e}{\partial a_2^e} & \cdots & \frac{\partial \mathcal{T}_1^e}{\partial a_{N+1}^e} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{\partial \mathcal{T}_{N+1}^e}{\partial b_0^e} & \frac{\partial \mathcal{T}_{N+1}^e}{\partial b_1^e} & \cdots & \frac{\partial \mathcal{T}_{N+1}^e}{\partial b_N^e} & \frac{\partial \mathcal{T}_{N+1}^e}{\partial a_1^e} & \frac{\partial \mathcal{T}_{N+1}^e}{\partial a_2^e} & \cdots & \frac{\partial \mathcal{T}_{N+1}^e}{\partial a_{N+1}^e} \\
\frac{\partial \mathcal{N}_1^e}{\partial b_0^e} & \frac{\partial \mathcal{N}_1^e}{\partial b_1^e} & \cdots & \frac{\partial \mathcal{N}_1^e}{\partial b_N^e} & \frac{\partial \mathcal{N}_1^e}{\partial a_1^e} & \frac{\partial \mathcal{N}_1^e}{\partial a_2^e} & \cdots & \frac{\partial \mathcal{N}_1^e}{\partial a_{N+1}^e} \\
\frac{\partial \mathcal{N}_2^e}{\partial b_0^e} & \frac{\partial \mathcal{N}_2^e}{\partial b_1^e} & \cdots & \frac{\partial \mathcal{N}_2^e}{\partial b_N^e} & \frac{\partial \mathcal{N}_2^e}{\partial a_1^e} & \frac{\partial \mathcal{N}_2^e}{\partial a_2^e} & \cdots & \frac{\partial \mathcal{N}_2^e}{\partial a_{N+1}^e} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{\partial \mathcal{N}_N^e}{\partial b_0^e} & \frac{\partial \mathcal{N}_N^e}{\partial b_1^e} & \cdots & \frac{\partial \mathcal{N}_N^e}{\partial b_N^e} & \frac{\partial \mathcal{N}_N^e}{\partial a_1^e} & \frac{\partial \mathcal{N}_N^e}{\partial a_2^e} & \cdots & \frac{\partial \mathcal{N}_N^e}{\partial a_{N+1}^e}
\end{vmatrix}$$

$$=\frac{(-1)^N}{(4\pi)^N}\left(\prod_{k=1}^{N+1}\frac{\partial\mathcal{T}_0^e}{\partial b_{k-1}^e}\frac{\partial\mathcal{T}_0^e}{\partial a_k^e}\right)\left(\prod_{j=1}^{N}\int_{a_j^e}^{b_j^e}(R_e(s_j))^{1/2}\,\mathrm{d}s_j\right)\Delta_d^e,$$

where

$$\Delta_d^e:=\begin{vmatrix} 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1\\ b_0^e & b_1^e & \cdots & b_N^e & a_1^e & a_2^e & \cdots & a_{N+1}^e\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ (b_0^e)^{N+1} & (b_1^e)^{N+1} & \cdots & (b_N^e)^{N+1} & (a_1^e)^{N+1} & (a_2^e)^{N+1} & \cdots & (a_{N+1}^e)^{N+1}\\ \frac{1}{s_1-b_0^e} & \frac{1}{s_1-b_1^e} & \cdots & \frac{1}{s_1-b_N^e} & \frac{1}{s_1-a_1^e} & \frac{1}{s_1-a_2^e} & \cdots & \frac{1}{s_1-a_{N+1}^e}\\ \frac{1}{s_2-b_0^e} & \frac{1}{s_2-b_1^e} & \cdots & \frac{1}{s_2-b_N^e} & \frac{1}{s_2-a_1^e} & \frac{1}{s_2-a_2^e} & \cdots & \frac{1}{s_2-a_{N+1}^e}\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ \frac{1}{s_N-b_0^e} & \frac{1}{s_N-b_1^e} & \cdots & \frac{1}{s_N-b_N^e} & \frac{1}{s_N-a_1^e} & \frac{1}{s_N-a_2^e} & \cdots & \frac{1}{s_N-a_{N+1}^e} \end{vmatrix}.$$

The above determinant, that is, Δ_d^e, has been calculated on pg. 780 of [92] (see, also, Section 5.3, Equations (5.148) and (5.149) of [62]), namely,

$$\Delta_d^e=\frac{\left(\prod_{j=1}^{N+1}\prod_{k=1}^{N+1}(b_{k-1}^e-a_j^e)\right)\left(\prod_{\substack{j,k=1\\ j<k}}^{N+1}(a_k^e-a_j^e)(b_{k-1}^e-b_{j-1}^e)\right)\left(\prod_{\substack{j,k=1\\ j<k}}^{N}(s_k-s_j)\right)}{(-1)^N\prod_{j=1}^{N}\prod_{k=1}^{N+1}(s_j-a_k^e)(s_j-b_{k-1}^e)};$$

but, for $-\infty<b_0^e<a_1^e<s_1<b_1^e<a_2^e<s_2<b_2^e<\cdots<b_{N-1}^e<a_N^e<s_N<b_N^e<a_{N+1}^e<+\infty$, $\Delta_d^e\neq 0$ (which means that it is of a fixed sign), and $\int_{a_j^e}^{b_j^e}(R_e(s_j))^{1/2}\,\mathrm{d}s_j>0$, $j=1,\dots,N$, whence

$$\left(\prod_{j=1}^{N}\int_{a_j^e}^{b_j^e}(R_e(s_j))^{1/2}\,\mathrm{d}s_j\right)\Delta_d^e\neq 0.$$

It remains to show that $\partial\mathcal{T}_0^e/\partial b_{k-1}^e$ and $\partial\mathcal{T}_0^e/\partial a_k^e$, $k=1,\dots,N+1$, too, are non-zero; for this purpose, one exploits the fact that $\mathcal{T}_0^e=(\mathrm{i}\pi)^{-1}\int_{J_e}(2s^{-1}+\widetilde{V}'(s))(R_e(s))_+^{-1/2}\,\mathrm{d}s$ is independent of z. It follows from Equation (3.7), the integral representation for $h_V^e(z)$ given in the Lemma, and Equations (F1) and (F2) that

$$\frac{(z-b_{k-1}^e)}{\sqrt{R_e(z)}}\frac{\partial\mathfrak{F}^e(z)}{\partial b_{k-1}^e}=-\frac{1}{\mathrm{i}\pi}\left((z-b_{k-1}^e)\frac{\partial h_V^e(z)}{\partial b_{k-1}^e}-\frac{1}{2}h_V^e(z)\right),\quad k=1,\dots,N+1,$$
$$\frac{(z-a_k^e)}{\sqrt{R_e(z)}}\frac{\partial\mathfrak{F}^e(z)}{\partial a_k^e}=-\frac{1}{\mathrm{i}\pi}\left((z-a_k^e)\frac{\partial h_V^e(z)}{\partial a_k^e}-\frac{1}{2}h_V^e(z)\right),\quad k=1,\dots,N+1:$$

using, now, the z-independence of $\mathcal{T}_0^e$, and the fact that, for the case of regular $\widetilde{V}\colon\mathbb{R}\setminus\{0\}\to\mathbb{R}$ satisfying conditions (2.3)–(2.5), $h_V^e(b_{j-1}^e), h_V^e(a_j^e)\neq 0$, $j=1,\dots,N+1$, one shows that

$$\left.\frac{(z-b_{k-1}^e)}{\sqrt{R_e(z)}}\frac{\partial\mathfrak{F}^e(z)}{\partial b_{k-1}^e}\right|_{z=b_{k-1}^e}=\frac{1}{2\pi\mathrm{i}}h_V^e(b_{k-1}^e)\neq 0,\quad k=1,\dots,N+1,$$
$$\left.\frac{(z-a_k^e)}{\sqrt{R_e(z)}}\frac{\partial\mathfrak{F}^e(z)}{\partial a_k^e}\right|_{z=a_k^e}=\frac{1}{2\pi\mathrm{i}}h_V^e(a_k^e)\neq 0,\quad k=1,\dots,N+1;$$

thus, via Equations (F1) and (F2), one arrives at

$$\frac{\partial\mathcal{T}_0^e}{\partial b_{k-1}^e}=-h_V^e(b_{k-1}^e)\neq 0\qquad\text{and}\qquad\frac{\partial\mathcal{T}_0^e}{\partial a_k^e}=-h_V^e(a_k^e)\neq 0,\quad k=1,\dots,N+1,$$

whence

$$\prod_{k=1}^{N+1}\frac{\partial\mathcal{T}_0^e}{\partial b_{k-1}^e}\frac{\partial\mathcal{T}_0^e}{\partial a_k^e}=\prod_{k=1}^{N+1}h_V^e(b_{k-1}^e)h_V^e(a_k^e)\neq 0.$$

Hence, $\mathrm{Jac}(\mathcal{T}_0^e,\dots,\mathcal{T}_{N+1}^e,\mathcal{N}_1^e,\dots,\mathcal{N}_N^e)\neq 0$.

It remains, still, to show that $\mathcal{T}_j^e$, $j=0,\dots,N+1$, and $\mathcal{N}_i^e$, $i=1,\dots,N$, are (real) analytic functions of $\{b_{j-1}^e,a_j^e\}_{j=1}^{N+1}$. From the definition of $\mathcal{T}_j^e$, $j\in\mathbb{Z}_0^+$, above, using the fact that they are independent of z, thus giving rise to zero residue contributions, a straightforward residue calculus calculation shows that, equivalently,

$$\mathcal{T}_j^e=\frac{1}{2}\oint_{C_{\mathrm{R}}^e}\left(\frac{2}{\mathrm{i}\pi s}+\frac{\widetilde{V}'(s)}{\mathrm{i}\pi}\right)\frac{s^j}{(R_e(s))^{1/2}}\,\mathrm{d}s,\quad j\in\mathbb{Z}_0^+,$$

where (the closed contour) C_{R}^e has been defined above: the only factor depending on $\{b_{k-1}^e,a_k^e\}_{k=1}^{N+1}$ is $\sqrt{R_e(z)}$. As $\sqrt{R_e(z)}$ is analytic $\forall\ z\in\mathbb{C}\setminus\cup_{j=1}^{N+1}[b_{j-1}^e,a_j^e]$, and since $C_{\mathrm{R}}^e\subset\mathbb{C}\setminus\cup_{j=1}^{N+1}[b_{j-1}^e,a_j^e]$, with $\mathrm{int}(C_{\mathrm{R}}^e)\supset\overline{J_e}\cup\{z\}$, it follows that, in particular, $\sqrt{R_e(z)}\restriction_{C_{\mathrm{R}}^e}$ is an analytic function of $\{b_{j-1}^e,a_j^e\}_{j=1}^{N+1}$, which implies, via the above (equivalent) contour integral representation of $\mathcal{T}_j^e$, $j\in\mathbb{Z}_0^+$, that $\mathcal{T}_k^e$, $k=0,\dots,N+1$, are (real) analytic functions of $\{b_{j-1}^e,a_j^e\}_{j=1}^{N+1}$. Recalling that $(\mathcal{H}\psi_V^e)(z)=\frac{1}{2\pi}(\frac{1}{z}+\frac{1}{2}\widetilde{V}'(z)-(R_e(z))^{1/2}h_V^e(z))$, it follows from the definition of $\mathcal{N}_j^e$, $j=1,\dots,N$, that

$$\mathcal{N}_j^e=-\frac{1}{2\pi}\int_{a_j^e}^{b_j^e}(R_e(s))^{1/2}h_V^e(s)\,\mathrm{d}s,\quad j=1,\dots,N:$$

making the linear change of variables $u_j\colon\mathbb{C}\to\mathbb{C}$, $s\mapsto u_j(s):=(b_j^e-a_j^e)^{-1}(s-a_j^e)$, $j=1,\dots,N$, which take each of the (compact) intervals $[a_j^e,b_j^e]$, $j=1,\dots,N$, onto $[0,1]$, and setting

$$\sqrt{\widehat{R}_e(z)}:=\left(\prod_{k_1=1}^{j}(z-b_{k_1-1}^e)\prod_{k_2=1}^{j-1}(z-a_{k_2}^e)\prod_{k_3=j+1}^{N+1}(a_{k_3}^e-z)\prod_{k_4=j+2}^{N+1}(b_{k_4-1}^e-z)\right)^{1/2},$$

one arrives at

$$\mathcal{N}_j^e=-\frac{1}{2\pi}\left(b_j^e-a_j^e\right)^2\int_0^1\left(u_j(1-u_j)\right)^{1/2}\left(\widehat{R}_e((b_j^e-a_j^e)u_j+a_j^e)\right)^{1/2}h_V^e((b_j^e-a_j^e)u_j+a_j^e)\,\mathrm{d}u_j,\quad j=1,\dots,N.$$

Recalling that $h_V^e(z)$ is analytic on $\mathbb{R}\setminus\{0\}$, in particular, $h_V^e(b_{j-1}^e),h_V^e(a_j^e)\neq 0$, $j=1,\dots,N+1$, and that it is an analytic function of $\{b_{k-1}^e(z_o),a_k^e(z_o)\}_{k=1}^{N+1}$ (since $-\infty<b_0^e<a_1^e<b_1^e<a_2^e<\cdots<b_N^e<a_{N+1}^e<+\infty$), and noting from the definition of $\sqrt{\widehat{R}_e(z)}$ above that, it, too, is an analytic function of $(b_j^e-a_j^e)u_j+a_j^e$, $(j,u_j)\in\{1,\dots,N\}\times[0,1]$, and thus an analytic function of $\{b_{j-1}^e(z_o),a_j^e(z_o)\}_{j=1}^{N+1}$, it follows that $\mathcal{N}_j^e$, $j=1,\dots,N$, are (real) analytic functions of $\{b_{j-1}^e(z_o),a_j^e(z_o)\}_{j=1}^{N+1}$.

Thus, as the Jacobian of the transformation $\{b_0^e(z_o),\dots,b_N^e(z_o),a_1^e(z_o),\dots,a_{N+1}^e(z_o)\}\mapsto\{\mathcal{T}_0^e,\dots,\mathcal{T}_{N+1}^e,\mathcal{N}_1^e,\dots,\mathcal{N}_N^e\}$ is non-zero whenever $\{b_{j-1}^e(z_o),a_j^e(z_o)\}_{j=1}^{N+1}$, the end-points of the support of the 'even' equilibrium measure, are chosen so that, for regular $\widetilde{V}\colon\mathbb{R}\setminus\{0\}\to\mathbb{R}$ satisfying conditions (2.3)–(2.5), $\overline{J_e}=\cup_{j=1}^{N+1}[b_{j-1}^e,a_j^e]$, and $\mathcal{T}_j^e$, $j=0,\dots,N+1$, and $\mathcal{N}_k^e$, $k=1,\dots,N$, are (real) analytic functions of $\{b_{j-1}^e(z_o),a_j^e(z_o)\}_{j=1}^{N+1}$, it follows, via the Implicit Function Theorem, that $b_{j-1}^e(z_o),a_j^e(z_o)$, $j=1,\dots,N+1$, are real analytic functions of z_o. □

Remark 3.2. It turns out that, for $\widetilde{V}\colon\mathbb{R}\setminus\{0\}\to\mathbb{R}$ (satisfying conditions (2.3)–(2.5)) of the form

$$\widetilde{V}(z)=\sum_{k=-2m_1}^{2m_2}\widetilde{\varrho}_k z^k,$$

with $\widetilde{\varrho}_k\in\mathbb{R}$, $k=-2m_1,\dots,2m_2$, $m_{1,2}\in\mathbb{N}$, and (since $\widetilde{V}(\pm\infty),\widetilde{V}(0)>0$) $\widetilde{\varrho}_{-2m_1},\widetilde{\varrho}_{2m_2}>0$, the integral for $h_V^e(z)$, that is, $h_V^e(z)=\frac{1}{2}\oint_{C_{\mathrm{R}}^e}(R_e(s))^{-1/2}(\frac{\mathrm{i}}{\pi s}+\frac{\mathrm{i}\widetilde{V}'(s)}{2\pi})(s-z)^{-1}\,\mathrm{d}s$, can be evaluated explicitly. Let $C_{\mathrm{R}}^e=\widetilde{\Gamma}_\infty^e\cup\widetilde{\Gamma}_0^e$, where $\widetilde{\Gamma}_\infty^e:=\{z'=R\mathrm{e}^{\mathrm{i}\vartheta},\ R>1/\varepsilon,\ \vartheta\in[0,2\pi]\}$ (oriented clockwise), and $\widetilde{\Gamma}_0^e:=\{z'=r\mathrm{e}^{\mathrm{i}\vartheta},\ 0<r<\varepsilon,\ \vartheta\in[0,2\pi]\}$ (oriented counter-clockwise), with ε some arbitrarily fixed, sufficiently small positive real number chosen such that: (i) $\partial\{z'\in\mathbb{C};\ |z'|=\varepsilon\}\cap\partial\{z'\in\mathbb{C};\ |z'|=1/\varepsilon\}=\varnothing$; (ii) $\{z'\in\mathbb{C};\ |z'|<\varepsilon\}\cap(J_e\cup\{z\})=\varnothing$; (iii)

$\{z' \in \mathbb{C};\ |z'| > 1/\varepsilon\} \cap (J_e \cup \{z\}) = \varnothing$; and (iv) $\{z' \in \mathbb{C};\ \varepsilon < |z'| < 1/\varepsilon\} \supset J_e \cup \{z\}$. A tedious, but otherwise straightforward, residue calculus calculation shows that

$$\begin{aligned}
h_V^e(z) &= \frac{1}{2} z^{2m_2-N-2} \sum_{j=0}^{2m_2-N-2} \sideset{}{'}\sum_{\substack{k_0,\dots,k_N \\ 0\leqslant|k|+|l|\leqslant 2m_2-j-N-2 \\ k_i\geqslant 0,\ l_i\geqslant 0,\ i\in\{0,\dots,N\}}} \sideset{}{'}\sum_{l_0,\dots,l_N} (2m_2-j)\widetilde{\varrho}_{2m_2-j}\left(\prod_{p=0}^{N}\prod_{j_p=0}^{k_p-1}\left(\frac{1}{2}+j_p\right)\right) \\
&\times\left(\prod_{q=0}^{N}\prod_{\widetilde{m}_q=0}^{l_q-1}\left(\frac{1}{2}+\widetilde{m}_q\right)\right)\frac{\left(\prod_{p'=0}^{N}(b_{p'}^e)^{k_{p'}}\right)\left(\prod_{q'=0}^{N}(a_{q'+1}^e)^{l_{q'}}\right)}{\left(\prod_{l'=0}^{N}k_{l'}!\right)\left(\prod_{\widetilde{m}'=0}^{N}l_{\widetilde{m}'}!\right)} z^{-(j+|k|+|l|)} \\
&+\frac{(-1)^{\mathcal{N}_+}\left(\prod_{k=1}^{N+1}|b_{k-1}^e a_k^e|\right)^{-1/2}}{2z^{2m_1+1}} \sum_{j=-2m_1+1}^{0} \sideset{}{''}\sum_{\substack{k_0,\dots,k_N \\ 0\leqslant|k|+|l|\leqslant 2m_1+j \\ k_i\geqslant 0,\ l_i\geqslant 0,\ i\in\{0,\dots,N\}}} \sideset{}{''}\sum_{l_0,\dots,l_N} (-2m_1-j)\widetilde{\varrho}_{-2m_1-j} \\
&\times\left(\prod_{p=0}^{N}\prod_{j_p=0}^{k_p-1}\left(\frac{1}{2}+j_p\right)\right)\left(\prod_{q=0}^{N}\prod_{\widetilde{m}_q=0}^{l_q-1}\left(\frac{1}{2}+\widetilde{m}_q\right)\right)\frac{\left(\prod_{p'=0}^{N}(b_{p'}^e)^{k_{p'}}\right)^{-1}\left(\prod_{q'=0}^{N}(a_{q'+1}^e)^{l_{q'}}\right)^{-1}}{\left(\prod_{l'=0}^{N}k_{l'}!\right)\left(\prod_{\widetilde{m}'=0}^{N}l_{\widetilde{m}'}!\right)} \\
&\times z^{|k|+|l|-j} + \frac{(-1)^{\mathcal{N}_+}\left(\prod_{k=1}^{N+1}|b_{k-1}^e a_k^e|\right)^{-1/2}}{z},
\end{aligned}$$

where $\mathcal{N}_+ \in \{0,\dots,N+1\}$ is the number of bands to the right of $z=0$, $|k| := k_0+k_1+\cdots+k_N$ $(\geqslant 0)$, $|l| := l_0+l_1+\cdots+l_N$ $(\geqslant 0)$, and the primes (resp., double primes) on the summations mean that all possible sums over $\{k_l\}_{l=0}^N$ and $\{l_k\}_{k=0}^N$ must be taken for which $0\leqslant k_0+\cdots+k_N+l_0+\cdots+l_N\leqslant 2m_2-j-N-2$, $j=0,\dots,2m_2-N-2$, $k_i\geqslant 0$, $l_i\geqslant 0$, $i=0,\dots,N$ (resp., $0\leqslant k_0+\cdots+k_N+l_0+\cdots+l_N\leqslant 2m_1+j$, $j=-2m_1+1,\dots,0$, $k_i\geqslant 0$, $l_i\geqslant 0$, $i=0,\dots,N$). It is important to note that all of the above sums are finite sums: any sums for which the upper limit is less than the lower limit are defined to be zero, and any products in which the upper limit is less than the lower limit are defined to be one; for example, $\sum_{j=0}^{-1}(*):=0$ and $\prod_{j=0}^{-1}(*):=1$.

It is also interesting to note that one may derive explicit formulae for the various moments of the 'even' equilibrium measure, that is, $\int_{J_e} s^{\pm m}\psi_V^e(s)\,\mathrm{d}s$, $m\in\mathbb{N}$, in terms of the external field and the function $(R_e(z))^{1/2}$; without loss of generality, and for demonstrative purposes only, consider, say, the following moments: $\int_{J_e} s^{\pm j}\,\mathrm{d}\mu_V^e(s)$, $j=1,2,3$ (the calculations below straightforwardly generalise to $\int_{J_e} s^{\pm(k+3)}\,\mathrm{d}\mu_V^e(s)$, $k\in\mathbb{N}$). Recall the following formulae for $\mathcal{F}^e(z)$ given in the proof of Lemma 3.5:

$$\mathcal{F}^e(z) = -\frac{1}{\pi\mathrm{i}z} - \frac{2}{\pi\mathrm{i}}\int_{J_e}\frac{\mathrm{d}\mu_V^e(s)}{s-z}, \quad z\in\mathbb{C}\setminus(J_e\cup\{0\}),$$

$$\mathcal{F}^e(z) = -\frac{1}{\pi\mathrm{i}z} - (R_e(z))^{1/2}\int_{J_e}\frac{\left(\frac{2\mathrm{i}}{\pi s}+\frac{\mathrm{i}\widetilde{V}'(s)}{\pi}\right)}{(R_e(s))_+^{1/2}(s-z)}\frac{\mathrm{d}s}{2\pi\mathrm{i}}, \quad z\in\mathbb{C}\setminus(J_e\cup\{0\}).$$

One derives the following asymptotic expansions: (1) for $\mu_V^e\in\mathcal{M}_1(\mathbb{R})$, in particular, $\int_{J_e} s^{-m}\,\mathrm{d}\mu_V^e(s)<\infty$, $m\in\mathbb{N}$, $s\in J_e$ and $z\notin J_e$, with $|z/s|\ll 1$ (e.g., $|z|\ll\min_{j=1,\dots,N+1}\{|b_{j-1}^e-a_j^e|\}$), via the expansions $\frac{1}{z-s} = -\sum_{k=0}^{l}\frac{z^k}{s^{k+1}}+\frac{z^{l+1}}{s^{l+1}(z-s)}$, $l\in\mathbb{Z}_0^+$, and $\ln(1-*) = -\sum_{k=1}^{\infty}\frac{*^k}{k}$, $|*|\ll 1$,

$$\mathcal{F}^e(z) \underset{z\to 0}{=} -\frac{1}{\pi\mathrm{i}z}-\frac{2}{\pi\mathrm{i}}\int_{J_e}s^{-1}\,\mathrm{d}\mu_V^e(s)+z\left(-\frac{2}{\pi\mathrm{i}}\int_{J_e}s^{-2}\,\mathrm{d}\mu_V^e(s)\right)+z^2\left(-\frac{2}{\pi\mathrm{i}}\int_{J_e}s^{-3}\,\mathrm{d}\mu_V^e(s)\right)+O(z^3),$$

and

$$\mathcal{F}^e(z) \underset{z\to 0}{=} -\frac{1}{\pi\mathrm{i}z}+\gamma_V^e\left(\check{Q}_0^e+z(\check{Q}_1^e-\check{P}_0^e\check{Q}_0^e)+z^2(\check{Q}_2^e-\check{P}_0^e\check{Q}_1^e+\check{P}_1^e\check{Q}_0^e)+O(z^3)\right),$$

where

$$\gamma_V^e := (-1)^{\mathcal{N}_+}\left(\prod_{j=1}^{N+1}\left|b_{j-1}^e a_j^e\right|\right)^{1/2}, \qquad \check{P}_0^e := \frac{1}{2}\sum_{j=1}^{N+1}\left(\frac{1}{b_{j-1}^e}+\frac{1}{a_j^e}\right),$$

$$\check{P}_1^e := \frac{1}{2}(\check{P}_0^e)^2 - \frac{1}{4}\sum_{j=1}^{N+1}\left(\frac{1}{(b_{j-1}^e)^2} + \frac{1}{(a_j^e)^2}\right), \qquad \check{Q}_j^e := -\int_{J_e} \frac{(\frac{2\mathrm{i}}{\pi s} + \frac{\mathrm{i}\widetilde{V}'(s)}{\pi})}{(R_e(s))_+^{1/2} s^{j+1}} \frac{\mathrm{d}s}{2\pi\mathrm{i}}, \quad j=0,1,2;$$

and (2) for $\mu_V^e \in \mathcal{M}_1(\mathbb{R})$, in particular, $\int_{J_e} \mathrm{d}\mu_V^e(s) = 1$ and $\int_{J_e} s^m \,\mathrm{d}\mu_V^e(s) < \infty$, $m \in \mathbb{N}$, $s \in J_e$ and $z \notin J_e$, with $|s/z| \ll 1$ (e.g., $|z| \gg \max_{j=1,\dots,N+1}\{|b_{j-1}^e - a_j^e|\}$), via the expansions $\frac{1}{s-z} = -\sum_{k=0}^{l} \frac{s^k}{z^{k+1}} + \frac{s^{l+1}}{z^{l+1}(s-z)}$, $l \in \mathbb{Z}_0^+$, and $\ln(1-*) = -\sum_{k=1}^{\infty} \frac{*^k}{k}$, $|*| \ll 1$,

$$\mathcal{F}^e(z) \underset{z\to\infty}{=} \frac{1}{\pi\mathrm{i}z} + \frac{1}{z^2}\left(\frac{2}{\pi\mathrm{i}}\int_{J_e} s\,\mathrm{d}\mu_V^e(s)\right) + \frac{1}{z^3}\left(\frac{2}{\pi\mathrm{i}}\int_{J_e} s^2\,\mathrm{d}\mu_V^e(s)\right) + \frac{1}{z^4}\left(\frac{2}{\pi\mathrm{i}}\int_{J_e} s^3\,\mathrm{d}\mu_V^e(s)\right) + \mathcal{O}(z^{-5}),$$

and

$$\mathcal{F}^e(z) \underset{z\to\infty}{=} -\frac{1}{\pi\mathrm{i}z} + z^N\left(1 - \frac{\alpha_V^e}{z} + \frac{\widetilde{P}_0^e}{z^2} + \frac{\widetilde{P}_1^e}{z^3} + \cdots\right)\int_{J_e} \frac{(\frac{2\mathrm{i}}{\pi s} + \frac{\mathrm{i}\widetilde{V}'(s)}{\pi})}{(R_e(s))_+^{1/2}}\left(1 + \cdots + \frac{s^N}{z^N} + \frac{s^{N+1}}{z^{N+1}} + \cdots\right)\frac{\mathrm{d}s}{2\pi\mathrm{i}},$$

where

$$\alpha_V^e := \frac{1}{2}\sum_{j=1}^{N+1}\left(b_{j-1}^e + a_j^e\right), \qquad \widetilde{P}_0^e := \frac{1}{2}(\alpha_V^e)^2 - \frac{1}{4}\sum_{j=1}^{N+1}\left((b_{j-1}^e)^2 + (a_j^e)^2\right),$$

$$\widetilde{P}_1^e := -\frac{1}{3!}\sum_{j=1}^{N+1}\left((b_{j-1}^e)^3 + (a_j^e)^3\right) - \frac{(\alpha_V^e)^3}{3!} + \frac{\alpha_V^e}{4}\sum_{j=1}^{N+1}\left((b_{j-1}^e)^2 + (a_j^e)^2\right).$$

Recalling the following (n-dependent) $N+2$ moment conditions stated in Lemma 3.5,

$$\int_{J_e} \frac{(\frac{2\mathrm{i}}{\pi s} + \frac{\mathrm{i}\widetilde{V}'(s)}{\pi})s^j}{(R_e(s))_+^{1/2}}\,\mathrm{d}s = 0, \quad j=0,\dots,N, \qquad \text{and} \qquad \int_{J_e} \frac{(\frac{2\mathrm{i}}{\pi s} + \frac{\mathrm{i}\widetilde{V}'(s)}{\pi})s^{N+1}}{(R_e(s))_+^{1/2}}\,\mathrm{d}s = 4,$$

and equating the respective pairs of asymptotic expansions above (as $z \to 0$ and $z \to \infty$) for $\mathcal{F}^e(z)$, one arrives at the following expressions for the first three 'positive' and 'negative' moments of the 'even' equilibrium measure:

$$\begin{aligned}
\int_{J_e} s\,\mathrm{d}\mu_V^e(s) &= \frac{1}{4}\int_{J_e} \frac{(\frac{2\mathrm{i}}{\pi s} + \frac{\mathrm{i}\widetilde{V}'(s)}{\pi})s^{N+2}}{(R_e(s))_+^{1/2}}\,\mathrm{d}s - \frac{1}{2}\sum_{j=1}^{N+1}(b_{j-1}^e + a_j^e), \\
\int_{J_e} s^2\,\mathrm{d}\mu_V^e(s) &= \frac{1}{4}\int_{J_e} \frac{(\frac{2\mathrm{i}}{\pi s} + \frac{\mathrm{i}\widetilde{V}'(s)}{\pi})s^{N+3}}{(R_e(s))_+^{1/2}}\,\mathrm{d}s - \frac{1}{8}\left(\sum_{j=1}^{N+1}(b_{j-1}^e + a_j^e)\right)\int_{J_e} \frac{(\frac{2\mathrm{i}}{\pi s} + \frac{\mathrm{i}\widetilde{V}'(s)}{\pi})s^{N+2}}{(R_e(s))_+^{1/2}}\,\mathrm{d}s \\
&\quad + \frac{1}{4}\left(\frac{1}{2}\left(\sum_{j=1}^{N+1}(b_{j-1}^e + a_j^e)\right)^2 - \sum_{j=1}^{N+1}((b_{j-1}^e)^2 + (a_j^e)^2)\right), \\
\int_{J_e} s^3\,\mathrm{d}\mu_V^e(s) &= \frac{1}{4}\int_{J_e} \frac{(\frac{2\mathrm{i}}{\pi s} + \frac{\mathrm{i}\widetilde{V}'(s)}{\pi})s^{N+4}}{(R_e(s))_+^{1/2}}\,\mathrm{d}s - \frac{1}{8}\left(\sum_{j=1}^{N+1}(b_{j-1}^e + a_j^e)\right)\int_{J_e} \frac{(\frac{2\mathrm{i}}{\pi s} + \frac{\mathrm{i}\widetilde{V}'(s)}{\pi})s^{N+3}}{(R_e(s))_+^{1/2}}\,\mathrm{d}s \\
&\quad + \frac{1}{16}\left(\frac{1}{2}\left(\sum_{j=1}^{N+1}(b_{j-1}^e + a_j^e)\right)^2 - \sum_{j=1}^{N+1}((b_{j-1}^e)^2 + (a_j^e)^2)\right)\int_{J_e} \frac{(\frac{2\mathrm{i}}{\pi s} + \frac{\mathrm{i}\widetilde{V}'(s)}{\pi})s^{N+2}}{(R_e(s))_+^{1/2}}\,\mathrm{d}s \\
&\quad - \frac{1}{8}\left(\frac{1}{3!}\left(\sum_{j=1}^{N+1}(b_{j-1}^e + a_j^e)\right)^3 + \frac{4}{3}\sum_{j=1}^{N+1}((b_{j-1}^e)^3 + (a_j^e)^3) - \sum_{j=1}^{N+1}(b_{j-1}^e + a_j^e)\right. \\
&\quad \left. \times \sum_{k=1}^{N+1}((b_{k-1}^e)^2 + (a_k^e)^2)\right), \\
\int_{J_e} s^{-1}\,\mathrm{d}\mu_V^e(s) &= \frac{1}{4}(-1)^{\mathcal{N}_+}\left(\prod_{j=1}^{N+1}|b_{j-1}^e a_j^e|\right)^{1/2}\int_{J_e} \frac{(\frac{2\mathrm{i}}{\pi s} + \frac{\mathrm{i}\widetilde{V}'(s)}{\pi})}{(R_e(s))_+^{1/2} s}\,\mathrm{d}s,
\end{aligned}$$

$$\int_{J_e} s^{-2}\,\mathrm{d}\mu_V^e(s)=\frac{1}{4}(-1)^{\mathcal{N}_+}\left(\prod_{j=1}^{N+1}|b_{j-1}^e a_j^e|\right)^{1/2}\left(\int_{J_e}\frac{(\frac{2\mathrm{i}}{\pi s}+\frac{\mathrm{i}\widetilde{V}'(s)}{\pi})}{(R_e(s))_+^{1/2}s^2}\,\mathrm{d}s-\frac{1}{2}\left(\sum_{j=1}^{N+1}\left(\frac{1}{b_{j-1}^e}+\frac{1}{a_j^e}\right)\right)\right.$$
$$\left.\times\int_{J_e}\frac{(\frac{2\mathrm{i}}{\pi s}+\frac{\mathrm{i}\widetilde{V}'(s)}{\pi})}{(R_e(s))_+^{1/2}s}\,\mathrm{d}s\right),$$
$$\int_{J_e} s^{-3}\,\mathrm{d}\mu_V^e(s)=\frac{1}{4}(-1)^{\mathcal{N}_+}\left(\prod_{j=1}^{N+1}|b_{j-1}^e a_j^e|\right)^{1/2}\left(\int_{J_e}\frac{(\frac{2\mathrm{i}}{\pi s}+\frac{\mathrm{i}\widetilde{V}'(s)}{\pi})}{(R_e(s))_+^{1/2}s^3}\,\mathrm{d}s-\frac{1}{2}\left(\sum_{j=1}^{N+1}\left(\frac{1}{b_{j-1}^e}+\frac{1}{a_j^e}\right)\right)\right.$$
$$\times\int_{J_e}\frac{(\frac{2\mathrm{i}}{\pi s}+\frac{\mathrm{i}\widetilde{V}'(s)}{\pi})}{(R_e(s))_+^{1/2}s^2}\,\mathrm{d}s+\left(\frac{1}{8}\left(\sum_{j=1}^{N+1}\left(\frac{1}{b_{j-1}^e}+\frac{1}{a_j^e}\right)\right)^2-\frac{1}{4}\sum_{j=1}^{N+1}\left(\frac{1}{(b_{j-1}^e)^2}+\frac{1}{(a_j^e)^2}\right)\right)$$
$$\left.\times\int_{J_e}\frac{(\frac{2\mathrm{i}}{\pi s}+\frac{\mathrm{i}\widetilde{V}'(s)}{\pi})}{(R_e(s))_+^{1/2}s}\,\mathrm{d}s\right).$$

It is important to note that all of the above integrals are real valued (since, for $s\in\overline{J_e}$, $(R_e(s))_+^{1/2}=\mathrm{i}(|R_e(s)|)^{1/2}\in\mathrm{i}\mathbb{R}$) and bounded (since, for $j=1,\dots,N+1$, $(R_e(s))^{1/2}=_{s\downarrow b_{j-1}^e}\mathcal{O}((s-b_{j-1}^e)^{1/2})$ and $(R_e(s))^{1/2}=_{s\uparrow a_j^e}\mathcal{O}((a_j^e-s)^{1/2})$, that is, there are removable singularities at the end-points of the support of the 'even' equilibrium measure). ■

Lemma 3.6. *Let the external field $\widetilde{V}\colon\mathbb{R}\setminus\{0\}\to\mathbb{R}$ satisfy conditions* (2.3)–(2.5). *Let the 'even' equilibrium measure, μ_V^e, and its support,* $\operatorname{supp}(\mu_V^e)=:J_e$ $(\subset\overline{\mathbb{R}}\setminus\{0,\pm\infty\})$, *be as described in Lemma* 3.5, *and let there exist ℓ_e $(\in\mathbb{R})$, the 'even' variational constant, such that*

$$\begin{gathered}4\int_{J_e}\ln(|x-s|)\psi_V^e(s)\,\mathrm{d}s-2\ln|x|-\widetilde{V}(x)-\ell_e=0,\quad x\in\overline{J_e},\\ 4\int_{J_e}\ln(|x-s|)\psi_V^e(s)\,\mathrm{d}s-2\ln|x|-\widetilde{V}(x)-\ell_e\leqslant 0,\quad x\in\mathbb{R}\setminus\overline{J_e},\end{gathered}\tag{3.9}$$

where, for $\widetilde{V}$ regular, the inequality in the second of Equations (3.9) *is strict. Then:*

(1) $g_+^e(z)+g_-^e(z)-\widetilde{V}(z)-\ell_e+2Q_e=0$, $z\in\overline{J_e}$, *where* $g_\pm^e(z):=\lim_{\varepsilon\downarrow 0}g^e(z\pm\mathrm{i}\varepsilon)$, *and Q_e is defined in Lemma* 3.4;

(2) $g_+^e(z)+g_-^e(z)-\widetilde{V}(z)-\ell_e+2Q_e\leqslant 0$, $z\in\mathbb{R}\setminus\overline{J_e}$, *where equality holds for at most a finite number of points, and, for $\widetilde{V}$ regular, the inequality is strict;*

(3) $g_+^e(z)-g_-^e(z)\in\mathrm{i}f_{g^e}^{\mathbb{R}}(z)$, $z\in\mathbb{R}$, *where $f_{g^e}^{\mathbb{R}}\colon\mathbb{R}\to\mathbb{R}$ is some bounded function, and, in particular,* $g_+^e(z)-g_-^e(z)=\mathrm{i}\,\mathrm{const.}$, $z\in\mathbb{R}\setminus\overline{J_e}$, *where* $\mathrm{const.}\in\mathbb{R}$;

(4) $\mathrm{i}(g_+^e(z)-g_-^e(z))'\geqslant 0$, $z\in J_e$, *and where, for $\widetilde{V}$ regular, equality holds for at most a finite number of points.*

Proof. Set (cf. Lemma 3.5) $J_e:=\cup_{j=1}^{N+1}J_j^e$, where $J_j^e=(b_{j-1}^e,a_j^e)=$ the jth 'band', with $N\in\mathbb{N}$ and finite, $b_0^e:=\min\{\operatorname{supp}(\mu_V^e)\}\notin\{-\infty,0\}$, $a_{N+1}^e:=\max\{\operatorname{supp}(\mu_V^e)\}\notin\{0,+\infty\}$, and $-\infty<b_0^e<a_1^e<b_1^e<a_2^e<\cdots<b_N^e<a_{N+1}^e<+\infty$, and $\{b_{j-1}^e,a_j^e\}_{j=1}^{N+1}$ satisfy the n-dependent and (locally) solvable system of $2(N+1)$ moment conditions given in Lemma 3.5. Consider the following cases: **(1)** $z\in\overline{J_j^e}:=[b_{j-1}^e,a_j^e]$, $j=1,\dots,N+1$; **(2)** $z\in(a_j^e,b_j^e)=$ the jth 'gap', $j=1,\dots,N$; **(3)** $z\in(a_{N+1}^e,+\infty)$; and **(4)** $z\in(-\infty,b_0^e)$.

(**1**) Recall the definition of $g^e(z)$ given in Lemma 3.4, namely, $g^e(z):=\int_{J_e}\ln((z-s)^2/zs)\psi_V^e(s)\,\mathrm{d}s$, $z\in\mathbb{C}\setminus(-\infty,\max\{0,a_{N+1}^e\})$, where the representation (cf. Lemma 3.5) $\mathrm{d}\mu_V^e(s)=\psi_V^e(s)\,\mathrm{d}s$, $s\in J_e$, was substituted into the latter formula. For $z\in\overline{J_j^e}$, $j=1,\dots,N+1$, one shows that

$$g_\pm^e(z)=2\int_{J_e}\ln(|z-s|)\psi_V^e(s)\,\mathrm{d}s\pm 2\pi\mathrm{i}\int_z^{a_{N+1}^e}\psi_V^e(s)\,\mathrm{d}s-Q_e-\begin{cases}\ln|z|, & z>0,\\ \ln|z|\pm\mathrm{i}\pi, & z<0,\end{cases}$$

where $g_\pm^e(z):=\lim_{\varepsilon\downarrow 0}g^e(z\pm\mathrm{i}\varepsilon)$, and $Q_e:=\int_{J_e}\ln(s)\psi_V^e(s)\,\mathrm{d}s$, whence

$$g_+^e(z)-g_-^e(z)=4\pi\mathrm{i}\int_z^{a_{N+1}^e}\psi_V^e(s)\,\mathrm{d}s+\begin{cases}0, & z>0,\\ -2\pi\mathrm{i}, & z<0,\end{cases}$$

which shows that $g^e_+(z)-g^e_-(z)\in \mathrm{i}\mathbb{R}$, and $\mathrm{Re}(g^e_+(z)-g^e_-(z))=0$; moreover, using the Fundamental Theorem of Calculus, one shows that $(g^e_+(z)-g^e_-(z))'=-4\pi\mathrm{i}\psi^e_V(z)$, whence $\mathrm{i}(g^e_+(z)-g^e_-(z))'=4\pi\psi^e_V(z)\geqslant 0$, since $\psi^e_V(z)\geqslant 0$ $\forall$ $z\in\overline{J_e}$ $(\supset \overline{J^e_j}$, $j=1,\dots,N+1)$. Furthermore, using the first of Equations (3.9), one shows that

$$g^e_+(z)+g^e_-(z)-\widetilde{V}(z)-l_e+2Q_e=4\int_{J_e}\ln(|z-s|)\psi^e_V(s)\,\mathrm{d}s-2\ln|z|-\widetilde{V}(z)-\ell_e=0,$$

which gives the formula for the 'even' variational constant ℓ_e $(\in\mathbb{R})$, which is the same [92, 97] (see, also, Section 7 of [56]) for each compact interval $\overline{J^e_j}$, $j=1,\dots,N+1$; in particular,

$$\ell_e=\frac{2}{\pi}\sum_{j=1}^{N+1}\int_{b^e_{j-1}}^{a^e_j}\ln\Big(\Big|\tfrac{1}{2}(b^e_N+a^e_{N+1})-s\Big|\Big)(|R_e(s)|)^{1/2}h^e_V(s)\,\mathrm{d}s-2\ln\Big|\tfrac{1}{2}(b^e_N+a^e_{N+1})\Big|-\widetilde{V}\Big(\tfrac{1}{2}(b^e_N+a^e_{N+1})\Big),$$

where $(|R_e(s)|)^{1/2}h^e_V(s)\geqslant 0$, $j=1,\dots,N+1$, and where there are no singularities in the integrand, since, for (any) $r>0$, $\lim_{|x|\to 0}|x|^r\ln|x|=0$.

(**2**) For $z\in(a^e_j,b^e_j)$, $j=1,\dots,N$, one shows that

$$g^e_\pm(z)=2\int_{J_e}\ln(|z-s|)\psi^e_V(s)\,\mathrm{d}s\pm 2\pi\mathrm{i}\sum_{k=j+1}^{N+1}\int_{b^e_{k-1}}^{a^e_k}\psi^e_V(s)\,\mathrm{d}s-Q_e-\begin{cases}\ln|z|, & z>0,\\ \ln|z|\pm\mathrm{i}\pi, & z<0,\end{cases}$$

whence

$$g^e_+(z)-g^e_-(z)=4\pi\mathrm{i}\int_{b^e_j}^{a^e_{N+1}}\psi^e_V(s)\,\mathrm{d}s+\begin{cases}0, & z>0,\\ -2\pi\mathrm{i}, & z<0,\end{cases}$$

which shows that $g^e_+(z)-g^e_-(z)=\mathrm{i}\,\mathrm{const.}$, with $\mathrm{const.}\in\mathbb{R}$, and $\mathrm{Re}(g^e_+(z)-g^e_-(z))=0$; moreover, $\mathrm{i}(g^e_+(z)-g^e_-(z))'=0$. One notes from the above formulae for $g^e_\pm(z)$ that

$$g^e_+(z)+g^e_-(z)-\widetilde{V}(z)-\ell_e+2Q_e=4\int_{J_e}\ln(|z-s|)\psi^e_V(s)\,\mathrm{d}s-2\ln|z|-\widetilde{V}(z)-\ell_e.$$

Recalling that (cf. Lemma 3.5) $\mathcal{H}\colon\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}\to\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}$, $f\mapsto(\mathcal{H}f)(z):=\fint_{\mathbb{R}}\frac{f(s)}{z-s}\frac{\mathrm{d}s}{\pi}$, where $\fint$ denotes the principle value integral, one shows that, for $z\in(a^e_j,b^e_j)$, $j=1,\dots,N$,

$$4\int_{J_e}\ln(|z-s|)\psi^e_V(s)\,\mathrm{d}s=4\pi\int_{a^e_j}^{z}(\mathcal{H}\psi^e_V)(s)\,\mathrm{d}s+4\int_{J_e}\ln(|a^e_j-s|)\psi^e_V(s)\,\mathrm{d}s;$$

thus,

$$\begin{aligned}g^e_+(z)+g^e_-(z)-\widetilde{V}(z)-\ell_e+2Q_e&=4\pi\int_{a^e_j}^{z}(\mathcal{H}\psi^e_V)(s)\,\mathrm{d}s+4\int_{J_e}\ln(|a^e_j-s|)\psi^e_V(s)\,\mathrm{d}s-2\ln|z|-\widetilde{V}(z)-\ell_e\\&=4\int_{J_e}\ln(|a^e_j-s|)\psi^e_V(s)\,\mathrm{d}s+4\pi\int_{a^e_j}^{z}(\mathcal{H}\psi^e_V)(s)\,\mathrm{d}s-4\pi\int_{a^e_j}^{z}\frac{\widetilde{V}'(s)}{4\pi}\,\mathrm{d}s\\&\quad-4\pi\int_{a^e_j}^{z}\frac{1}{2\pi s}\,\mathrm{d}s-2\ln|a^e_j|-\widetilde{V}(a^e_j)-\ell_e\\&=4\pi\int_{a^e_j}^{z}\left((\mathcal{H}\psi^e_V)(s)-\frac{\widetilde{V}'(s)}{4\pi}-\frac{1}{2\pi s}\right)\mathrm{d}s\\&\quad+\underbrace{\left(4\int_{J_e}\ln(|a^e_j-s|)\psi^e_V(s)\,\mathrm{d}s-2\ln|a^e_j|-\widetilde{V}(a^e_j)-\ell_e\right)}_{=0}\quad\Rightarrow\end{aligned}$$

$$g^e_+(z)+g^e_-(z)-\widetilde{V}(z)-\ell_e+2Q_e=4\pi\int_{a^e_j}^{z}\left((\mathcal{H}\psi^e_V)(s)-\frac{\widetilde{V}'(s)}{4\pi}-\frac{1}{2\pi s}\right)\mathrm{d}s,\quad z\in(a^e_j,b^e_j),\quad j=1,\dots,N.$$

It was shown in the proof of Lemma 3.5 that $(\mathcal{H}\psi_V^e)(s) = \frac{\widetilde{V}'(s)}{4\pi} + \frac{1}{2\pi s} - \frac{1}{2\pi}(R_e(s))^{1/2}h_V^e(s)$, $s \in (a_j^e, b_j^e)$, $j=1,\dots,N$, whence

$$g_+^e(z)+g_-^e(z)-\widetilde{V}(z)-\ell_e+2Q_e=-2\int_{a_j^e}^{z}(R_e(s))^{1/2}h_V^e(s)\,\mathrm{d}s<0, \quad z\in\cup_{j=1}^{N}(a_j^e,b_j^e):$$

since $h_V^e(z)$ is real analytic on $\mathbb{R}\setminus\{0\}$ and $(R_e(s))^{1/2}h_V^e(s)>0$ $\forall$ $s\in\cup_{j=1}^{N}(a_j^e,b_j^e)$, it follows that one has equality only at points $z\in\cup_{j=1}^{N}(a_j^e,b_j^e)$ for which $h_V^e(z)=0$, which are finitely denumerable. (Note that, for $z\in\cup_{j=1}^{N}(a_j^e,b_j^e)$, $(R_e(s))_+^{1/2}=(R_e(s))_-^{1/2}=(R_e(s))^{1/2}$.)

(**3**) For $z\in(a_{N+1}^e,+\infty)$, one shows that

$$g_\pm^e(z)=2\int_{J_e}\ln(|z-s|)\psi_V^e(s)\,\mathrm{d}s-Q_e-\begin{cases}\ln|z|, & z>0,\\ \ln|z|\pm\mathrm{i}\pi, & z<0,\end{cases}$$

whence

$$g_+^e(z)-g_-^e(z)=\begin{cases}0, & z>0,\\ -2\pi\mathrm{i}, & z<0,\end{cases}$$

which shows that $g_+^e(z)-g_-^e(z)$ is pure imaginary, and $\mathrm{i}(g_+^e(z)-g_-^e(z))'=0$. Also, one shows that

$$g_+^e(z)+g_-^e(z)-\widetilde{V}(z)-\ell_e+2Q_e=4\int_{J_e}\ln(|z-s|)\psi_V^e(s)\,\mathrm{d}s-2\ln|z|-\widetilde{V}(z)-\ell_e;$$

and, following the analysis of case (**2**) above, one shows that, for $z\in(a_{N+1}^e,+\infty)$,

$$4\int_{J_e}\ln(|z-s|)\psi_V^e(s)\,\mathrm{d}s-2\ln|z|-\widetilde{V}(z)-\ell_e=4\pi\int_{a_{N+1}^e}^{z}\left((\mathcal{H}\psi_V^e)(s)-\frac{\widetilde{V}'(s)}{4\pi}-\frac{1}{2\pi s}\right)\mathrm{d}s,$$

thus, via the relation (cf. case (**2**) above) $(\mathcal{H}\psi_V^e)(s)=\frac{\widetilde{V}'(s)}{4\pi}+\frac{1}{2\pi s}-\frac{1}{2\pi}(R_e(s))^{1/2}h_V^e(s)$, $s\in(a_{N+1}^e,+\infty)$, one arrives at

$$g_+^e(z)+g_-^e(z)-\widetilde{V}(z)-\ell_e+2Q_e=-2\int_{a_{N+1}^e}^{z}(R_e(s))^{1/2}h_V^e(s)\,\mathrm{d}s<0, \quad z\in(a_{N+1}^e,+\infty).$$

If: (1) $z\to+\infty$ (e.g., $|z|\gg\max_{j=1,\dots,N+1}\{|b_{j-1}^e,a_j^e|\}$), $s\in J_e$, and $|s/z|\ll1$, from $\mu_V^e\in\mathcal{M}_1(\mathbb{R})$, in particular, $\int_{J_e}\mathrm{d}\mu_V^e(s)=1$ and $\int_{J_e}s^m\,\mathrm{d}\mu_V^e(s)<\infty$, $m\in\mathbb{N}$, the formula for $g_+^e(z)+g_-^e(z)-\widetilde{V}(z)-\ell_e+2Q_e$ above, and the expansions $\frac{1}{s-z}=-\sum_{k=0}^{l}\frac{s^k}{z^{k+1}}+\frac{s^{l+1}}{z^{l+1}(s-z)}$, $l\in\mathbb{Z}_0^+$, and $\ln(z-s)=_{|z|\to\infty}\ln(z)-\sum_{k=1}^{\infty}\frac{1}{k}(\frac{s}{z})^k$, one shows that

$$g_+^e(z)+g_-^e(z)-\widetilde{V}(z)-\ell_e+2Q_e\underset{z\to+\infty}{=}\ln(z^2+1)-\widetilde{V}(z)+\mathcal{O}(1),$$

which, upon recalling that (cf. condition (2.4)) $\lim_{|x|\to\infty}(\widetilde{V}(x)/\ln(x^2+1))=+\infty$, shows that $g_+^e(z)+g_-^e(z)-\widetilde{V}(z)-\ell_e+2Q_e<0$; and (2) $|z|\to0$ (e.g., $|z|\ll\min_{j=1,\dots,N+1}\{|b_{j-1}^e,a_j^e|\}$), $s\in J_e$, and $|z/s|\ll1$, from $\mu_V^e\in\mathcal{M}_1(\mathbb{R})$, in particular, $\int_{J_e}s^{-m}\,\mathrm{d}\mu_V^e(s)<\infty$, $m\in\mathbb{N}$, the above formula for $g_+^e(z)+g_-^e(z)-\widetilde{V}(z)-\ell_e+2Q_e$, and the expansions $\frac{1}{z-s}=-\sum_{k=0}^{l}\frac{z^k}{s^{k+1}}+\frac{z^{l+1}}{s^{l+1}(z-s)}$, $l\in\mathbb{Z}_0^+$, and $\ln(s-z)=_{|z|\to0}\ln(s)-\sum_{k=1}^{\infty}\frac{1}{k}(\frac{z}{s})^k$, one shows that

$$g_+^e(z)+g_-^e(z)-\widetilde{V}(z)-\ell_e+2Q_e\underset{|z|\to0}{=}\ln(z^{-2}+1)-\widetilde{V}(z)+\mathcal{O}(1),$$

whereupon, recalling that (cf. condition (2.5)) $\lim_{|x|\to0}(\widetilde{V}(x)/\ln(x^{-2}+1))=+\infty$, it follows that $g_+^e(z)+g_-^e(z)-\widetilde{V}(z)-\ell_e+2Q_e<0$.

(**4**) For $z\in(-\infty,b_0^e)$, one shows that

$$g_\pm^e(z)=2\int_{J_e}\ln(|z-s|)\psi_V^e(s)\,\mathrm{d}s\pm2\pi\mathrm{i}-Q_e-\begin{cases}\ln|z|, & z>0,\\ \ln|z|\pm\mathrm{i}\pi, & z<0,\end{cases}$$

whence

$$g^e_+(z)-g^e_-(z)=4\pi\mathrm{i}+\begin{cases}0, & z>0,\\ -2\pi\mathrm{i}, & z<0,\end{cases}$$

which shows that $g^e_+(z)-g^e_-(z)$ is pure imaginary, and $\mathrm{i}(g^e_+(z)-g^e_-(z))'=0$. Also,

$$g^e_+(z)+g^e_-(z)-\widetilde{V}(z)-\ell_e+2Q_e=4\int_{J_e}\ln(|z-s|)\psi^e_V(s)\,\mathrm{d}s-2\ln|z|-\widetilde{V}(z)-\ell_e:$$

proceeding as in the asymptotic analysis for case (**3**) above, one arrives at

$$g^e_+(z)+g^e_-(z)-\widetilde{V}(z)-\ell_e+2Q_e\underset{z\to-\infty}{=}\ln(z^2+1)-\widetilde{V}(z)+\mathcal{O}(1),$$

and

$$g^e_+(z)+g^e_-(z)-\widetilde{V}(z)-\ell_e+2Q_e\underset{|z|\to0}{=}\ln(z^{-2}+1)-\widetilde{V}(z)+\mathcal{O}(1),$$

whence, via conditions (2.4) and (2.5), $g^e_+(z)+g^e_-(z)-\widetilde{V}(z)-\ell_e+2Q_e<0$, $z\in(-\infty,b^e_0)$. □

4 The Model RHP and Parametrices

In this section, the (normalised at infinity) auxiliary RHP for $\overset{e}{\mathcal{M}}\colon\mathbb{C}\setminus\mathbb{R}\to\mathrm{SL}_2(\mathbb{C})$ formulated in Lemma 3.4 is augmented, by means of a sequence of contour deformations and transformations *à la* Deift-Venakides-Zhou [1–3], into simpler, 'model' RHPs which, as $n\to\infty$, are solved explicitly (in closed form) in terms of Riemann theta functions (associated with the underlying genus-N hyperelliptic Riemann surface) and Airy functions, and which give rise to the leading ($\mathcal{O}(1)$) terms of asymptotics for $\boldsymbol{\pi}_{2n}(z)$, $\xi^{(2n)}_n$ and $\phi_{2n}(z)$ stated, respectively, in Theorems 2.3.1 and 2.3.2, and the asymptotic (as $n\to\infty$) analysis of the parametrices, which are 'approximate' solutions of the RHP for $\overset{e}{\mathcal{M}}\colon\mathbb{C}\setminus\mathbb{R}\to\mathrm{SL}_2(\mathbb{C})$ in neighbourhoods of the end-points of the support of the 'even' equilibrium measure, and which give rise to the $\mathcal{O}(n^{-1})$ (and $\mathcal{O}(n^{-2})$) corrections for $\boldsymbol{\pi}_{2n}(z)$, $\xi^{(2n)}_n$ and $\phi_{2n}(z)$ stated, respectively, in Theorems 2.3.1 and 2.3.2, is undertaken.

Lemma 4.1. *Let the external field $\widetilde{V}\colon\mathbb{R}\setminus\{0\}\to\mathbb{R}$ satisfy conditions* (2.3)–(2.5)*; furthermore, let $\widetilde{V}$ be regular. Let the 'even' equilibrium measure, μ^e_V, and its support,* $\operatorname{supp}(\mu^e_V)=:J_e=\cup_{j=1}^{N+1}J^e_j:=\cup_{j=1}^{N+1}(b^e_{j-1},a^e_j)$*, be as described in Lemma* 3.5*, and, along with ℓ_e ($\in\mathbb{R}$), the 'even' variational constant, satisfy the variational conditions stated in Lemma* 3.6*, Equations* (3.9)*; moreover, let conditions* (1)–(4) *stated in Lemma* 3.6 *be valid. Then the* RHP *for $\overset{e}{\mathcal{M}}\colon\mathbb{C}\setminus\mathbb{R}\to\mathrm{SL}_2(\mathbb{C})$ formulated in Lemma* 3.4 *can be equivalently reformulated as follows:* (1) *$\overset{e}{\mathcal{M}}(z)$ is holomorphic for $z\in\mathbb{C}\setminus\mathbb{R}$;* (2) *$\overset{e}{\mathcal{M}}_\pm(z):=\lim_{\substack{z'\to z\\ \pm\mathrm{Im}(z')>0}}\overset{e}{\mathcal{M}}(z')$ satisfy the boundary condition*

$$\overset{e}{\mathcal{M}}_+(z)=\overset{e}{\mathcal{M}}_-(z)\overset{e}{\upsilon}(z),\quad z\in\mathbb{R},$$

where

$$\overset{e}{\upsilon}(z)=\begin{cases}\begin{pmatrix}\mathrm{e}^{-4n\pi\mathrm{i}\int_z^{a^e_{N+1}}\psi^e_V(s)\,\mathrm{d}s} & 1\\ 0 & \mathrm{e}^{4n\pi\mathrm{i}\int_z^{a^e_{N+1}}\psi^e_V(s)\,\mathrm{d}s}\end{pmatrix}, & z\in(b^e_{j-1},a^e_j),\quad j=1,\dots,N+1,\\ \begin{pmatrix}\mathrm{e}^{-4n\pi\mathrm{i}\int_{b^e_k}^{a^e_{N+1}}\psi^e_V(s)\,\mathrm{d}s} & \mathrm{e}^{n(g^e_+(z)+g^e_-(z)-\widetilde{V}(z)-\ell_e+2Q_e)}\\ 0 & \mathrm{e}^{4n\pi\mathrm{i}\int_{b^e_k}^{a^e_{N+1}}\psi^e_V(s)\,\mathrm{d}s}\end{pmatrix}, & z\in(a^e_k,b^e_k),\quad k=1,\dots,N,\\ \mathrm{I}+\mathrm{e}^{n(g^e_+(z)+g^e_-(z)-\widetilde{V}(z)-\ell_e+2Q_e)}\sigma_+, & z\in(-\infty,b^e_0)\cup(a^e_{N+1},+\infty),\end{cases}$$

with $g^e(z)$ and Q_e defined in Lemma 3.4*,*

$$\pm\mathrm{Re}\left(\mathrm{i}\int_z^{a^e_{N+1}}\psi^e_V(s)\,\mathrm{d}s\right)>0,\quad z\in\mathbb{C}_\pm\cap(\cup_{j=1}^{N+1}\mathbb{U}^e_j),$$

where $\mathbb{U}_j^e := \{z \in \mathbb{C}^*;\ \mathrm{Re}(z) \in (b_{j-1}^e, a_j^e),\ \inf_{q \in J_j^e} |z-q| < r_j \in (0,1)\}$, $j=1,\dots,N+1$, *with* $\mathbb{U}_i^e \cap \mathbb{U}_j^e = \varnothing$, $i \neq j = 1,\dots,N+1$, *and* $g_+^e(z)+g_-^e(z)-\widetilde{V}(z)-\ell_e+2Q_e<0$, $z \in (-\infty,b_0^e) \cup (a_{N+1}^e,+\infty) \cup (\cup_{j=1}^N (a_j^e,b_j^e))$; (3) $\overset{e}{\mathcal{M}}(z) \underset{\substack{z\to\infty \\ z\in\mathbb{C}\setminus\mathbb{R}}}{=} \mathrm{I}+\mathcal{O}(z^{-1})$; *and* (4) $\overset{e}{\mathcal{M}}(z) \underset{\substack{z\to 0 \\ z\in\mathbb{C}\setminus\mathbb{R}}}{=} \mathcal{O}(1)$.

Proof. Item (1) stated in the Lemma is simply a re-statement of item (1) of Lemma 3.4. Write $\mathbb{R}=(-\infty,b_0^e)\cup(a_{N+1}^e,+\infty)\cup(\cup_{j=1}^{N+1}J_j^e)\cup(\cup_{k=1}^N(a_k^e,b_k^e))\cup(\cup_{l=1}^{N+1}\{b_{l-1}^e,a_l^e\})$, where $J_j^e := (b_{j-1}^e,a_j^e)$, $j=1,\dots,N+1$. Recall from the proof of Lemma 3.6 that, for $\widetilde{V}$, μ_V^e, and ℓ_e described therein (and in the Lemma): (1)

$$g_+^e(z)-g_-^e(z)=\begin{cases} 4\pi\mathrm{i}\int_z^{a_{N+1}^e}\psi_V^e(s)\,\mathrm{d}s+\begin{cases}0, & z\in\mathbb{R}_+\cap\overline{J_j^e}, \quad j=1,\dots,N+1,\\ -2\pi\mathrm{i}, & z\in\mathbb{R}_-\cap\overline{J_j^e}, \quad j=1,\dots,N+1,\end{cases}\\ 4\pi\mathrm{i}\int_{b_j^e}^{a_{N+1}^e}\psi_V^e(s)\,\mathrm{d}s+\begin{cases}0, & z\in\mathbb{R}_+\cap(a_j^e,b_j^e), \quad j=1,\dots,N,\\ -2\pi\mathrm{i}, & z\in\mathbb{R}_-\cap(a_j^e,b_j^e), \quad j=1,\dots,N,\end{cases}\\ \begin{cases}0, & z\in\mathbb{R}_+\cap(a_{N+1}^e,+\infty),\\ -2\pi\mathrm{i}, & z\in\mathbb{R}_-\cap(a_{N+1}^e,+\infty),\end{cases}\\ 4\pi\mathrm{i}+\begin{cases}0, & z\in\mathbb{R}_+\cap(-\infty,b_0^e),\\ -2\pi\mathrm{i}, & z\in\mathbb{R}_-\cap(-\infty,b_0^e),\end{cases}\end{cases}$$

where $\overline{J_j^e}\ (:=J_j^e\cup\partial J_j^e)=[b_{j-1}^e,a_j^e]$, $j=1,\dots,N+1$; and (2)

$$g_+^e(z)+g_-^e(z)-\widetilde{V}(z)-\ell_e+2Q_e=\begin{cases}0, & z\in\cup_{j=1}^{N+1}\overline{J_j^e},\\ -2\int_{a_j^e}^z(R_e(s))^{1/2}h_V^e(s)\,\mathrm{d}s<0, & z\in(a_j^e,b_j^e), \quad j=1,\dots,N,\\ -2\int_{a_{N+1}^e}^z(R_e(s))^{1/2}h_V^e(s)\,\mathrm{d}s<0, & z\in(a_{N+1}^e,+\infty),\\ 2\int_z^{b_0^e}(R_e(s))^{1/2}h_V^e(s)\,\mathrm{d}s<0, & z\in(-\infty,b_0^e).\end{cases}$$

Recall, also, the formula for the 'jump matrix' given in Lemma 3.4, namely,

$$\begin{pmatrix}\mathrm{e}^{-n(g_+^e(z)-g_-^e(z))} & \mathrm{e}^{n(g_+^e(z)+g_-^e(z)-\widetilde{V}(z)-\ell_e+2Q_e)}\\ 0 & \mathrm{e}^{n(g_+^e(z)-g_-^e(z))}\end{pmatrix}.$$

Partitioning $\mathbb{R}$ as given above, one obtains the formula for $\overset{e}{\upsilon}(z)$ stated in the Lemma, thus item (2); moreover, items (3) and (4) are re-statements of the respective items of Lemma 3.4. It remains, therefore, to show that $\mathrm{Re}(\mathrm{i}\int_z^{a_{N+1}^e}\psi_V^e(s)\,\mathrm{d}s)$ satisfies the inequalities stated in the Lemma. Recall from the proof of Lemma 3.4 that $g^e(z)$ is uniformly Lipschitz continuous in $\mathbb{C}_\pm$; moreover, via the Cauchy-Riemann conditions, item (4) of Lemma 3.6, that is, $\mathrm{i}(g_+^e(z)-g_-^e(z))' \geqslant 0$, $z\in J_e$, implies that the quantity $g_+^e(z)-g_-^e(z)$ has an analytic continuation, $\mathcal{G}^e(z)$, say, to an open neighbourhood, $\mathbb{U}_V^e$, say, of $J_e=\cup_{j=1}^{N+1}(b_{j-1}^e,a_j^e)$, where $\mathbb{U}_V^e := \cup_{j=1}^{N+1}\mathbb{U}_j^e$, with $\mathbb{U}_j^e=\{z\in\mathbb{C}^*;\ \mathrm{Re}(z)\in(b_{j-1}^e,a_j^e),\ \inf_{q\in J_j^e}|z-q|<r_j\in(0,1)\}$, $j=1,\dots,N+1$, and $\mathbb{U}_i^e\cap\mathbb{U}_j^e=\varnothing$, $i\neq j=1,\dots,N+1$, with the property that $\pm\mathrm{Re}(\mathcal{G}^e(z))>0$ for $z\in\mathbb{C}_\pm\cap\mathbb{U}_V^e$. □

Remark 4.1. Recalling that the external field $\widetilde{V}\colon \mathbb{R}\setminus\{0\}\to\mathbb{R}$ is regular, that is, $h_V^e(z)\not\equiv 0\ \forall\ z\in\overline{J_j^e} := \cup_{j=1}^{N+1}[b_{j-1}^e,a_j^e]$, the second inequality of Equations (3.9) is strict, namely, $4\int_{J_e}\ln(|x-s|)\psi_V^e(s)\,\mathrm{d}s-2\ln|x|-\widetilde{V}(x)-\ell_e+2Q_e<0$, $x\in\mathbb{R}\setminus\overline{J_e}$, and (from the proof of Lemma 4.1) that $g_+^e(z)+g_-^e(z)-\widetilde{V}(z)-\ell_e+2Q_e<0$, $z\in(-\infty,b_0^e)\cup(a_{N+1}^e,+\infty)\cup(\cup_{j=1}^N(a_j^e,b_j^e))$, it follows that

$$\overset{e}{\upsilon}(z) \underset{n\to\infty}{=} \begin{cases}\mathrm{e}^{-(4n\pi\mathrm{i}\int_{b_j^e}^{a_{N+1}^e}\psi_V^e(s)\,\mathrm{d}s)\sigma_3}(\mathrm{I}+o(1)\sigma_+), & z\in(a_j^e,b_j^e), \quad j=1,\dots,N,\\ \mathrm{I}+o(1)\sigma_+, & z\in(-\infty,b_0^e)\cup(a_{N+1}^e,+\infty),\end{cases}$$

where $o(1)$ denotes terms that are exponentially small. ■

Lemma 4.2. *Let the external field* $\widetilde{V}\colon\mathbb{R}\setminus\{0\}\to\mathbb{R}$ *satisfy conditions* (2.3)–(2.5); *furthermore, let* $\widetilde{V}$ *be regular. Let the 'even' equilibrium measure,* μ_V^e*, and its support,* $\mathrm{supp}(\mu_V^e)=: J_e=\cup_{j=1}^{N+1}J_j^e := \cup_{j=1}^{N+1}(b_{j-1}^e,a_j^e)$*, be as*

described in Lemma 3.5*, and, along with* ℓ_{e} $(\in \mathbb{R})$*, the 'even' variational constant, satisfy the variational conditions stated in Lemma* 3.6*, Equations* (3.9)*; moreover, let conditions* (1)–(4) *stated in Lemma* 3.6 *be valid. Let* $\overset{e}{\mathcal{M}} \colon \mathbb{C} \setminus \mathbb{R} \to \mathrm{SL}_{2}(\mathbb{C})$ *solve the* RHP *formulated in Lemma* 4.1*, and let the deformed (and oriented) contour* $\Sigma_{e}^{\sharp} := \mathbb{R} \cup (\cup_{j=1}^{N+1}(J_{j}^{e,\frown} \cup J_{j}^{e,\smile}))$ *be as in Figure* 8 *below; furthermore,* $\cup_{j=1}^{N+1}(\Omega_{j}^{e,\frown} \cup \Omega_{j}^{e,\smile} \cup J_{j}^{e,\frown} \cup J_{j}^{e,\smile}) \subset \cup_{j=1}^{N+1}\mathbb{U}_{j}^{e}$ *(Figure* 8*), where* $\mathbb{U}_{j}^{e}$*,* $j=1,\dots,N+1$*, are defined in Lemma* 4.1*. Set*

$$\overset{e}{\mathcal{M}}{}^{\sharp}(z) := \begin{cases} \overset{e}{\mathcal{M}}(z), & z \in \mathbb{C} \setminus (\Sigma_{e}^{\sharp} \cup (\cup_{j=1}^{N+1}(\Omega_{j}^{e,\frown} \cup \Omega_{j}^{e,\smile}))), \\ \overset{e}{\mathcal{M}}(z)\Big(\mathrm{I} - \mathrm{e}^{-4n\pi\mathrm{i}\int_{z}^{a_{N+1}^{e}} \psi_{V}^{e}(s)\,\mathrm{d}s}\,\sigma_{-}\Big), & z \in \mathbb{C}_{+} \cap (\cup_{j=1}^{N+1}\Omega_{j}^{e,\frown}), \\ \overset{e}{\mathcal{M}}(z)\Big(\mathrm{I} + \mathrm{e}^{4n\pi\mathrm{i}\int_{z}^{a_{N+1}^{e}} \psi_{V}^{e}(s)\,\mathrm{d}s}\,\sigma_{-}\Big), & z \in \mathbb{C}_{-} \cap (\cup_{j=1}^{N+1}\Omega_{j}^{e,\smile}). \end{cases}$$

Then $\overset{e}{\mathcal{M}}{}^{\sharp} \colon \mathbb{C} \setminus \Sigma_{e}^{\sharp} \to \mathrm{SL}_{2}(\mathbb{C})$ *solves the following, equivalent* RHP: (1) $\overset{e}{\mathcal{M}}{}^{\sharp}(z)$ *is holomorphic for* $z \in \mathbb{C} \setminus \Sigma_{e}^{\sharp}$; (2) $\overset{e}{\mathcal{M}}{}^{\sharp}_{\pm}(z) := \lim_{\substack{z' \to z \\ z' \in \pm\,\mathrm{side\,of}\,\Sigma_{e}^{\sharp}}} \overset{e}{\mathcal{M}}{}^{\sharp}(z')$ *satisfy the boundary condition*

$$\overset{e}{\mathcal{M}}{}^{\sharp}_{+}(z) = \overset{e}{\mathcal{M}}{}^{\sharp}_{-}(z)\overset{e}{\upsilon}{}^{\sharp}(z), \quad z \in \Sigma_{e}^{\sharp},$$

where

$$\overset{e}{\upsilon}{}^{\sharp}(z) = \begin{cases} \mathrm{i}\sigma_{2}, & z \in J_{j}^{e}, \quad j=1,\dots,N+1, \\ \mathrm{I} + \mathrm{e}^{-4n\pi\mathrm{i}\int_{z}^{a_{N+1}^{e}} \psi_{V}^{e}(s)\,\mathrm{d}s}\,\sigma_{-}, & z \in J_{j}^{e,\frown}, \quad j=1,\dots,N+1, \\ \mathrm{I} + \mathrm{e}^{4n\pi\mathrm{i}\int_{z}^{a_{N+1}^{e}} \psi_{V}^{e}(s)\,\mathrm{d}s}\,\sigma_{-}, & z \in J_{j}^{e,\smile}, \quad j=1,\dots,N+1, \\ \begin{pmatrix} \mathrm{e}^{-4n\pi\mathrm{i}\int_{b_{k}^{e}}^{a_{N+1}^{e}} \psi_{V}^{e}(s)\,\mathrm{d}s} & \mathrm{e}^{n(g_{+}^{e}(z)+g_{-}^{e}(z)-\widetilde{V}(z)-\ell_{e}+2Q_{e})} \\ 0 & \mathrm{e}^{4n\pi\mathrm{i}\int_{b_{k}^{e}}^{a_{N+1}^{e}} \psi_{V}^{e}(s)\,\mathrm{d}s} \end{pmatrix}, & z \in (a_{k}^{e}, b_{k}^{e}), \quad k=1,\dots,N, \\ \mathrm{I} + \mathrm{e}^{n(g_{+}^{e}(z)+g_{-}^{e}(z)-\widetilde{V}(z)-\ell_{e}+2Q_{e})}\sigma_{+}, & z \in (-\infty, b_{0}^{e}) \cup (a_{N+1}^{e}, +\infty), \end{cases}$$

with $\mathrm{Re}(\mathrm{i}\int_{z}^{a_{N+1}^{e}} \psi_{V}^{e}(s)\,\mathrm{d}s) > 0$ *(resp.,* $\mathrm{Re}(\mathrm{i}\int_{z}^{a_{N+1}^{e}} \psi_{V}^{e}(s)\,\mathrm{d}s) < 0$*),* $z \in \mathbb{C}_{+} \cap \Omega_{j}^{e,\frown}$ *(resp.,* $z \in \mathbb{C}_{-} \cap \Omega_{j}^{e,\smile}$*),* $j=1,\dots,N+1$; (3)

$$\overset{e}{\mathcal{M}}{}^{\sharp}(z) \underset{\substack{z \to \infty \\ z \in \mathbb{C} \setminus (\Sigma_{e}^{\sharp} \cup (\cup_{j=1}^{N+1}(\Omega_{j}^{e,\frown} \cup \Omega_{j}^{e,\smile})))}}{=} \mathrm{I} + \mathcal{O}(z^{-1});$$

and (4)

$$\overset{e}{\mathcal{M}}{}^{\sharp}(z) \underset{\substack{z \to 0 \\ z \in \mathbb{C} \setminus (\Sigma_{e}^{\sharp} \cup (\cup_{j=1}^{N+1}(\Omega_{j}^{e,\frown} \cup \Omega_{j}^{e,\smile})))}}{=} \mathcal{O}(1).$$

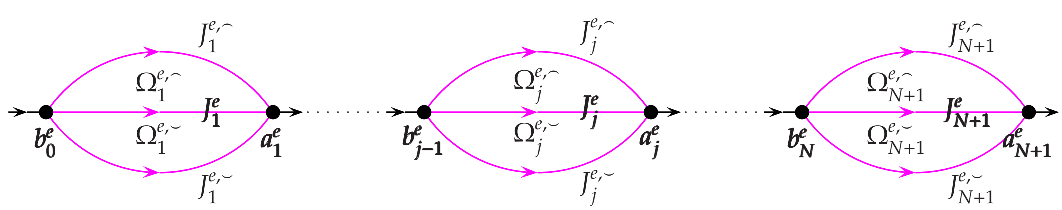

Figure 8: Oriented/deformed contour $\Sigma_{e}^{\sharp} := \mathbb{R} \cup (\cup_{j=1}^{N+1}(J_{j}^{e,\frown} \cup J_{j}^{e,\smile}))$

Proof. Items (1), (3), and (4) in the formulation of the RHP for $\overset{e}{\mathcal{M}}{}^{\sharp} \colon \mathbb{C} \setminus \Sigma_{e}^{\sharp} \to \mathrm{SL}_{2}(\mathbb{C})$ follow from the definition of $\overset{e}{\mathcal{M}}{}^{\sharp}(z)$ (in terms of $\overset{e}{\mathcal{M}}(z)$) given in the Lemma and the respective items (1), (3), and (4) for the RHP for $\overset{e}{\mathcal{M}} \colon \mathbb{C} \setminus \mathbb{R} \to \mathrm{SL}_{2}(\mathbb{C})$ stated in Lemma 4.1; it remains, therefore, to verify item (2), that

is, the formula for $\overset{e}{v}{}^{\sharp}(z)$. Recall from item (2) of Lemma 4.1 that, for $z \in (b_{j-1}^{e}, a_{j}^{e})$ $(\subset J_{e})$, $j=1,\dots,N+1$, $\overset{e}{\mathcal{M}}_{+}(z)=\overset{e}{\mathcal{M}}_{-}(z)\overset{e}{v}(z)$, where $\overset{e}{v}(z)=\begin{pmatrix} \mathrm{e}^{-4n\pi \mathrm{i}\int_{z}^{a_{N+1}^{e}}\psi_{V}^{e}(s)\,\mathrm{d}s} & 1 \\ 0 & \mathrm{e}^{4n\pi \mathrm{i}\int_{z}^{a_{N+1}^{e}}\psi_{V}^{e}(s)\,\mathrm{d}s} \end{pmatrix}$: noting the matrix factorisation

$$\begin{pmatrix} \mathrm{e}^{-4n\pi \mathrm{i}\int_{z}^{a_{N+1}^{e}}\psi_{V}^{e}(s)\,\mathrm{d}s} & 1 \\ 0 & \mathrm{e}^{4n\pi \mathrm{i}\int_{z}^{a_{N+1}^{e}}\psi_{V}^{e}(s)\,\mathrm{d}s} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \mathrm{e}^{4n\pi \mathrm{i}\int_{z}^{a_{N+1}^{e}}\psi_{V}^{e}(s)\,\mathrm{d}s} & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ \mathrm{e}^{-4n\pi \mathrm{i}\int_{z}^{a_{N+1}^{e}}\psi_{V}^{e}(s)\,\mathrm{d}s} & 1 \end{pmatrix},$$

it follows that, for $z \in (b_{j-1}^{e}, a_{j}^{e})$, $j=1,\dots,N+1$,

$$\overset{e}{\mathcal{M}}_{+}(z)\begin{pmatrix} 1 & 0 \\ -\mathrm{e}^{-4n\pi \mathrm{i}\int_{z}^{a_{N+1}^{e}}\psi_{V}^{e}(s)\,\mathrm{d}s} & 1 \end{pmatrix} = \overset{e}{\mathcal{M}}_{-}(z)\begin{pmatrix} 1 & 0 \\ \mathrm{e}^{4n\pi \mathrm{i}\int_{z}^{a_{N+1}^{e}}\psi_{V}^{e}(s)\,\mathrm{d}s} & 1 \end{pmatrix}\mathrm{i}\sigma_{2}.$$

It was shown in Lemma 4.1 that $\pm\mathrm{Re}(\mathrm{i}\int_{z}^{a_{N+1}^{e}}\psi_{V}^{e}(s)\,\mathrm{d}s)>0$ for $z \in \mathbb{C}_{\pm} \cap \mathbb{U}_{j}^{e}$, where $\mathbb{U}_{j}^{e}:=\{z \in \mathbb{C}^{*};\ \mathrm{Re}(z) \in (b_{j-1}^{e}, a_{j}^{e}),\ \inf_{q \in J_{j}^{e}}|z-q| < r_{j} \in (0,1)\}$, $j=1,\dots,N+1$, with $\mathbb{U}_{i}^{e} \cap \mathbb{U}_{j}^{e}=\varnothing$, $i \neq j=1,\dots,N+1$, and $J_{j}^{e}:=(b_{j-1}^{e}, a_{j}^{e})$, $j=1,\dots,N+1$. (One notes that the terms $\pm 4n\pi\mathrm{i}\int_{z}^{a_{N+1}^{e}}\psi_{V}^{e}(s)\,\mathrm{d}s$, which are pure imaginary for $z \in \mathbb{R}$, and corresponding to which $\exp(\pm 4n\pi\mathrm{i}\int_{z}^{a_{N+1}^{e}}\psi_{V}^{e}(s)\,\mathrm{d}s)$ are undulatory, are continued analytically to $\mathbb{C}_{\pm} \cap (\cup_{j=1}^{N+1}\mathbb{U}_{j}^{e})$, respectively, corresponding to which $\exp(\pm 4n\pi\mathrm{i}\int_{z}^{a_{N+1}^{e}}\psi_{V}^{e}(s)\,\mathrm{d}s)$ are exponentially decreasing as $n \to \infty$). As per the DZ non-linear steepest-descent method [1,2] (see, also, the extension [3]), one now 'deforms' the original (and oriented) contour $\mathbb{R}$ to the deformed, or extended, (and oriented) contour/skeleton $\Sigma_{e}^{\sharp}:=\mathbb{R} \cup (\cup_{j=1}^{N+1}(J_{j}^{e,\frown} \cup J_{j}^{e,\smile}))$ (Figure 8) in such a way that the upper (resp., lower) 'lips' of the 'lenses' $J_{j}^{e,\frown}$ (resp., $J_{j}^{e,\smile}$), $j=1,\dots,N+1$, which are the boundaries of $\Omega_{j}^{e,\frown}$ (resp., $\Omega_{j}^{e,\smile}$), $j=1,\dots,N+1$, respectively, lie within the domain of analytic continuation of $g_{+}^{e}(z)-g_{-}^{e}(z)$ (cf. the proof of Lemma 4.1), that is, $\cup_{j=1}^{N+1}(\Omega_{j}^{e,\frown} \cup \Omega_{j}^{e,\smile} \cup J_{j}^{e,\frown} \cup J_{j}^{e,\smile}) \subset \cup_{j=1}^{N+1}\mathbb{U}_{j}^{e}$; in particular, each (oriented) interval $J_{j}^{e}=(b_{j-1}^{e}, a_{j}^{e})$, $j=1,\dots,N+1$, in the original (and oriented) contour $\mathbb{R}$ is 'split' (or branched) into three, and the new (and oriented) contour $\Sigma_{e}^{\sharp}$ is the old contour ($\mathbb{R}$) together with the (oriented) boundary of $N+1$ lens-shaped regions, one region surrounding each (bounded and oriented) interval J_{j}^{e}. Now, recalling the definition of $\overset{e}{\mathcal{M}}{}^{\sharp}(z)$ (in terms of $\overset{e}{\mathcal{M}}(z)$) stated in the Lemma, and the expression for (the jump matrix) $\overset{e}{v}(z)$ given in Lemma 4.1, one arrives at the formula for $\overset{e}{v}{}^{\sharp}(z)$ given in item (2) of the Lemma. □

Remark 4.2. The jump condition stated in item (2) of Lemma 4.2, that is, $\overset{e}{\mathcal{M}}{}^{\sharp}_{+}(z)=\overset{e}{\mathcal{M}}{}^{\sharp}_{-}(z)\overset{e}{v}{}^{\sharp}(z)$, $z \in \Sigma_{e}^{\sharp}$, with $\overset{e}{v}{}^{\sharp}(z)$ given therein, should, of course, be understood as follows: the $\mathrm{SL}_{2}(\mathbb{C})$-valued functions $\overset{e}{\mathcal{M}}{}^{\sharp}\restriction_{\mathbb{C}_{\pm}\setminus\Sigma_{e}^{\sharp}}$ have a continuous extension to $\Sigma_{e}^{\sharp}$ with boundary values $\overset{e}{\mathcal{M}}{}^{\sharp}_{\pm}(z):=\lim_{\substack{z' \to z \in \Sigma_{e}^{\sharp} \\ z' \in \pm \text{ side of } \Sigma_{e}^{\sharp}}}\overset{e}{\mathcal{M}}{}^{\sharp}(z')$ satisfying the above jump relation ($\overset{e}{\mathcal{M}}{}^{\sharp}(z)$ is continuous in each component of $\mathbb{C} \setminus \Sigma_{e}^{\sharp}$ up to the boundary with boundary values $\overset{e}{\mathcal{M}}{}^{\sharp}_{\pm}(z)$ satisfying the above jump relation on $\Sigma_{e}^{\sharp}$). ■

Recalling from Lemma 4.1 that, for $z \in (-\infty, b_{0}^{e}) \cup (a_{N+1}^{e}, +\infty) \cup (\cup_{j=1}^{N}(a_{j}^{e}, b_{j}^{e}))$, $g_{+}^{e}(z)+g_{-}^{e}(z)-\widetilde{V}(z)-\ell_{e}+2Q_{e}<0$, and, from Lemma 4.2, $\mathrm{Re}(\mathrm{i}\int_{z}^{a_{N+1}^{e}}\psi_{V}^{e}(s)\,\mathrm{d}s) > 0$ for $z \in J_{j}^{e,\frown}$ (resp., $\mathrm{Re}(\mathrm{i}\int_{z}^{a_{N+1}^{e}}\psi_{V}^{e}(s)\,\mathrm{d}s) < 0$ for $z \in J_{j}^{e,\smile}$), $j=1,\dots,N+1$, one arrives at the following large-n asymptotic behaviour for

the jump matrix $\overset{e}{v}{}^{\sharp}(z)$:

$$\overset{e}{v}{}^{\sharp}(z) \underset{n\to\infty}{=} \begin{cases} \mathrm{i}\sigma_2, & z \in J_j^e, \quad j=1,\dots,N+1, \\ \mathrm{I}+\mathcal{O}(\mathrm{e}^{-nc|z|})\sigma_-, & z \in J_j^{e,\frown} \cup J_j^{e,\smile}, \quad j=1,\dots,N+1, \\ \mathrm{e}^{-(4n\pi\mathrm{i}\int_{b_j^e}^{a_{N+1}^e}\psi_V^e(s)\,\mathrm{d}s)\sigma_3}\Big(\mathrm{I}+\mathcal{O}(\mathrm{e}^{-nc|z-a_j^e|})\sigma_+\Big), & z \in (a_j^e,b_j^e) \setminus \widehat{\mathbb{U}}_{\delta_0^e}(0), \quad j=1,\dots,N, \\ \mathrm{e}^{-(4n\pi\mathrm{i}\int_{b_j^e}^{a_{N+1}^e}\psi_V^e(s)\,\mathrm{d}s)\sigma_3}\Big(\mathrm{I}+\mathcal{O}(\mathrm{e}^{-nc|z|^{-1}})\sigma_+\Big), & z \in (a_j^e,b_j^e) \cap \widehat{\mathbb{U}}_{\delta_0^e}(0), \quad j=1,\dots,N, \\ \mathrm{I}+\mathcal{O}(\mathrm{e}^{-nc|z|})\sigma_+, & z \in ((-\infty,b_0^e)\cup(a_{N+1}^e,+\infty)) \setminus \widehat{\mathbb{U}}_{\delta_0^e}(0), \\ \mathrm{I}+\mathcal{O}(\mathrm{e}^{-nc|z|^{-1}})\sigma_+, & z \in ((-\infty,b_0^e)\cup(a_{N+1}^e,+\infty)) \cap \widehat{\mathbb{U}}_{\delta_0^e}(0), \end{cases}$$

where c (some generic number) >0, $\widehat{\mathbb{U}}_{\delta_0^e}(0) := \{z \in \mathbb{C};\, |z| < \delta_0^e\}$, with δ_0^e some arbitrarily fixed, sufficiently small positive real number, and where the respective convergences are normal, that is, uniform in (respective) compact subsets (see Section 5 below).

Recall from Lemma 2.56 of [1] that, for an oriented skeleton in $\mathbb{C}$ on which the jump matrix of an RHP is defined, one may always choose to add or delete a portion of the skeleton on which the jump matrix equals I without altering the RHP in the operator sense; hence, neglecting those jumps on $\Sigma_e^{\sharp}$ tending exponentially quickly (as $n \to \infty$) to I, and removing the corresponding oriented skeletons from $\Sigma_e^{\sharp}$, it becomes more or less transparent how to construct a parametrix, that is, an approximate solution, of the (normalised at infinity) RHP for $\overset{e}{\mathcal{M}}{}^{\sharp} \colon \mathbb{C} \setminus \Sigma_e^{\sharp} \to \mathrm{SL}_2(\mathbb{C})$ stated in Lemma 4.2, namely, the large-n solution of the RHP for $\overset{e}{\mathcal{M}}{}^{\sharp}(z)$ formulated in Lemma 4.2 should be 'close to' the solution of the following (normalised at infinity) limiting, or model, RHP (for $\overset{e}{m}{}^{\infty}(z)$).

Lemma 4.3. *Let the external field $\widetilde{V} \colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ satisfy conditions* (2.3)–(2.5); *furthermore, let $\widetilde{V}$ be regular. Let the 'even' equilibrium measure, μ_V^e, and its support,* $\operatorname{supp}(\mu_V^e) =: J_e = \cup_{j=1}^{N+1} J_j^e := \cup_{j=1}^{N+1}(b_{j-1}^e, a_j^e)$*, be as described in Lemma* 3.5*, and, along with ℓ_e ($\in \mathbb{R}$), the 'even' variational constant, satisfy the variational conditions stated in Lemma* 3.6*, Equations* (3.9); *moreover, let conditions* (1)–(4) *stated in Lemma* 3.6 *be valid. Then $\overset{e}{m}{}^{\infty} \colon \mathbb{C} \setminus J_e^{\infty} \to \mathrm{SL}_2(\mathbb{C})$, where $J_e^{\infty} := J_e \cup (\cup_{j=1}^{N}(a_j^e, b_j^e))$, solves the following (model)* RHP: (1) *$\overset{e}{m}{}^{\infty}(z)$ is holomorphic for $z \in \mathbb{C} \setminus J_e^{\infty}$;* (2) *$\overset{e}{m}{}^{\infty}_{\pm}(z) := \lim_{\substack{z' \to z \\ z' \in \pm \text{ side of } J_e^{\infty}}} \overset{e}{m}{}^{\infty}(z')$ satisfy the boundary condition*

$$\overset{e}{m}{}^{\infty}_{+}(z) = \overset{e}{m}{}^{\infty}_{-}(z)\overset{e}{v}{}^{\infty}(z), \quad z \in J_e^{\infty},$$

where

$$\overset{e}{v}{}^{\infty}(z) = \begin{cases} \mathrm{i}\sigma_2, & z \in (b_{j-1}^e, a_j^e), \quad j=1,\dots,N+1, \\ \mathrm{e}^{-(4n\pi\mathrm{i}\int_{b_j^e}^{a_{N+1}^e}\psi_V^e(s)\,\mathrm{d}s)\sigma_3}, & z \in (a_j^e, b_j^e), \quad j=1,\dots,N; \end{cases}$$

(3) $\overset{e}{m}{}^{\infty}(z) \underset{\substack{z\to\infty \\ z\in\mathbb{C}\setminus J_e^{\infty}}}{=} \mathrm{I}+\mathcal{O}(z^{-1})$; *and* (4) $\overset{e}{m}{}^{\infty}(z) \underset{\substack{z\to 0 \\ z\in\mathbb{C}\setminus J_e^{\infty}}}{=} \mathcal{O}(1)$.

The model RHP for $\overset{e}{m}{}^{\infty} \colon \mathbb{C} \setminus J_e^{\infty} \to \mathrm{SL}_2(\mathbb{C})$ formulated in Lemma 4.3 is (explicitly) solvable in terms of Riemann theta functions (see, for example, Section 3 of [57]; see, also, Section 4.2 of [58]): the solution is succinctly presented below.

Lemma 4.4. *Let $\gamma^e \colon \mathbb{C} \setminus ((-\infty, b_0^e) \cup (a_{N+1}^e, +\infty) \cup (\cup_{j=1}^{N}(a_j^e, b_j^e))) \to \mathbb{C}$ be defined by*

$$\gamma^e(z) := \begin{cases} \left(\left(\dfrac{z-b_0^e}{z-a_{N+1}^e}\right)\displaystyle\prod_{k=1}^{N}\left(\dfrac{z-b_k^e}{z-a_k^e}\right)\right)^{1/4}, & z \in \mathbb{C}_+, \\ -\mathrm{i}\left(\left(\dfrac{z-b_0^e}{z-a_{N+1}^e}\right)\displaystyle\prod_{k=1}^{N}\left(\dfrac{z-b_k^e}{z-a_k^e}\right)\right)^{1/4}, & z \in \mathbb{C}_-. \end{cases}$$

Then, on the lower edge of each finite-length gap, that is, $(a_j^e, b_j^e)^-$, $j=1,\dots,N$, $\gamma^e(z) + (\gamma^e(z))^{-1}$ has exactly one root/zero, denoted $\Big\{z_j^{e,-} \in (a_j^e, b_j^e)^- \subset \mathbb{C}_-,\ j=1,\dots,N;\ (\gamma^e(z)+(\gamma^e(z))^{-1})|_{z=z_j^{e,-}} = 0\Big\}$, and, on the upper

edge of each finite-length gap, that is, $(a_j^e, b_j^e)^+$, $j=1,\dots,N$, $\gamma^e(z)-(\gamma^e(z))^{-1}$ *has exactly one root/zero, denoted* $\left\{z_j^{e,+} \in (a_j^e, b_j^e)^+ \subset \mathbb{C}_+,\ j=1,\dots,N;\ (\gamma^e(z)-(\gamma^e(z))^{-1})\vert_{z=z_j^{e,+}}=0\right\}$ *(in the plane,* $z_j^{e,+}=z_j^{e,-}:=z_j^e \in (a_j^e, b_j^e)$, $j \in 1,\dots,N$*). Furthermore,* $\gamma^e(z)$ *solves the following (scalar)* RHP:

(1) $\gamma^e(z)$ *is holomorphic for* $z \in \mathbb{C} \setminus ((-\infty, b_0^e) \cup (a_{N+1}^e, +\infty) \cup (\cup_{j=1}^N (a_j^e, b_j^e)))$;

(2) $\gamma_+^e(z)=\gamma_-^e(z)\mathrm{i}$, $z \in (-\infty, b_0^e) \cup (a_{N+1}^e, +\infty) \cup (\cup_{j=1}^N (a_j^e, b_j^e))$;

(3) $\gamma^e(z) \underset{\substack{z \to \infty \\ z \in \mathbb{C}_\pm}}{=} (-\mathrm{i})^{(1 \mp 1)/2}(1+\mathcal{O}(z^{-1}))$; *and*

(4) $\gamma^e(z) \underset{\substack{z \to 0 \\ z \in \mathbb{C}_\pm}}{=} \mathcal{O}(1)$.

Proof. Define $\gamma^e(z)$ as in the Lemma. Then, one notes that $\gamma^e(z) \mp (\gamma^e(z))^{-1}=0 \Leftrightarrow (\gamma^e(z))^2 \mp 1 = 0 \Rightarrow (\gamma^e(z))^4-1=0 \Leftrightarrow \mathcal{Q}^e(z)\ (\in \mathbb{R}[z]) := (z-b_0^e)\prod_{k=1}^N(z-b_k^e)-(z-a_{N+1}^e)\prod_{k=1}^N(z-a_k^e)=0$, whence, via a straightforward calculation, one shows that $\mathcal{Q}^e(a_j^e)=(-1)^{N-j+1}\widehat{\mathcal{Q}}_{a_j^e}^e$, $j=1,\dots,N$, where $\widehat{\mathcal{Q}}_{a_j^e}^e := (b_j^e-a_j^e)(a_j^e-b_0^e)\prod_{k=1}^{j-1}(a_j^e-b_k^e)\prod_{l=j+1}^N(b_l^e-a_j^e)\ (>0)$, and $\mathcal{Q}^e(b_j^e)=-(-1)^{N-j+1}\widehat{\mathcal{Q}}_{b_j^e}^e$, $j=1,\dots,N$, where $\widehat{\mathcal{Q}}_{b_j^e}^e := (b_j^e-a_j^e)(a_{N+1}^e-b_j^e)\prod_{k=1}^{j-1}(b_j^e-a_k^e)\prod_{l=j+1}^N(a_l^e-b_j^e)\ (>0)$; thus, $\mathcal{Q}^e(a_j^e)\mathcal{Q}^e(b_j^e)<0$, $j=1,\dots,N$, which shows that: (i) $\mathcal{Q}^e(z)$ has a root/zero, z_j^e, in each (open) interval (a_j^e, b_j^e), $j=1,\dots,N$; and (ii) since $\mathcal{Q}^e(z)$ is a unital polynomial with $\deg(\mathcal{Q}^e(z))=N$, $\{z_j^e\}_{j=1}^N$ are the only (simple) zeros/roots of $\mathcal{Q}^e(z)$. A straightforward analysis of the branch cuts shows that, for $z \in \cup_{j=1}^N (a_j^e, b_j^e)^\pm$, $\pm(\gamma^e(z))^2>0$, whence $\{z_j^{e,\pm}\}_{j=1}^N=\{z^\pm \in (a_j^e, b_j^e)^\pm \subset \mathbb{C}_\pm,\ j=1,\dots,N;\ (\gamma^e(z) \mp (\gamma^e(z))^{-1})\vert_{z=z^\pm}=0\}$. Setting $\widetilde{J}^e := (-\infty, b_0^e) \cup (a_{N+1}^e, +\infty) \cup (\cup_{j=1}^N (a_j^e, b_j^e))$, one shows that, upon performing a straightforward analysis of the branch cuts, $\gamma^e(z)$ solves the RHP $(\gamma^e(z), \mathrm{i}, \widetilde{J}^e)$ formulated in the Lemma. □

Remark 4.3. Recall from the proof of Lemma 4.4 that $\mathcal{Q}^e(z) := (z-b_0^e)\prod_{k=1}^N(z-b_k^e)-(z-a_{N+1}^e)\prod_{k=1}^N(z-a_k^e)\ (\in \mathbb{R}[z])$, with $\deg(\mathcal{Q}^e(z))=N$; furthermore, $\mathcal{Q}^e(z_j^e)=0$, where $z_j^e \in (a_j^e, b_j^e)$, $j=1,\dots,N$. Writing $\mathcal{Q}^e(z)=\sum_{j=0}^N \mathfrak{q}_j^e z^j$, where, in particular, $\mathfrak{q}_0^e=(-1)^N(\prod_{j=1}^{N+1} a_j^e-\prod_{l=1}^{N+1} b_{l-1}^e)$ and $\mathfrak{q}_N^e=\sum_{j=1}^{N+1}(a_j^e-b_{j-1}^e)\ (\neq 0)$, one uses a particular case of *Gerschgorin's Circle Theorem* to arrive at the following (upper) bound for the roots/zeros z_j^e, $j=1,\dots,N$: $\vert z_j^e\vert \leqslant \vert \mathfrak{q}_N^e\vert^{-1}\sum_{l=0}^N \vert \mathfrak{q}_l^e\vert$, $j=1,\dots,N$. ■

All of the notation/nomenclature used in Lemma 4.5 below has been defined at the end of Subsection 2.1; the reader, therefore, is advised to peruse the relevant notations(s), etc., before proceeding to Lemma 4.5. Let $\mathcal{Y}_e$ denote the Riemann surface of $y^2=R_e(z)=\prod_{k=1}^{N+1}(z-b_{k-1}^e)(z-a_k^e)$, where the single-valued branch of the square root is chosen so that $z^{-(N+1)}(R_e(z))^{1/2} \underset{\substack{z \to \infty \\ z \in \mathbb{C}_\pm}}{\sim} \pm 1$. Let $\mathcal{P} := (y, z)$ denote a point on the Riemann surface $\mathcal{Y}_e$ $(:=\{(y,z);\ y^2=R_e(z)\})$. The notation $\infty^\pm$ means: $\mathcal{P} \to \infty^\pm \Leftrightarrow z \to \infty$, $y \sim \pm z^{N+1}$.

Lemma 4.5. *Let* $\overset{e}{m}{}^\infty : \mathbb{C} \setminus J_e^\infty \to \mathrm{SL}_2(\mathbb{C})$ *solve the* RHP *formulated in Lemma* 4.3. *Then,*

$$\overset{e}{m}{}^\infty(z)=\begin{cases} \overset{e}{\mathfrak{M}}{}^\infty(z), & z \in \mathbb{C}_+, \\ -\mathrm{i}\,\overset{e}{\mathfrak{M}}{}^\infty(z)\sigma_2, & z \in \mathbb{C}_-, \end{cases}$$

where

$$\overset{e}{\mathfrak{M}}{}^\infty(z) := \begin{pmatrix} \frac{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\infty)+\boldsymbol{d}_e)}{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\infty)-\frac{n}{2\pi}\boldsymbol{\Omega}^e+\boldsymbol{d}_e)} & 0 \\ 0 & \frac{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\infty)+\boldsymbol{d}_e)}{\boldsymbol{\theta}^e(-\boldsymbol{u}_+^e(\infty)-\frac{n}{2\pi}\boldsymbol{\Omega}^e-\boldsymbol{d}_e)} \end{pmatrix} \overset{e}{\boldsymbol{\Theta}}{}^\infty(z),$$

and

$$\overset{e}{\boldsymbol{\Theta}}{}^\infty(z)=\begin{pmatrix} \frac{(\gamma^e(z)+(\gamma^e(z))^{-1})}{2}\frac{\boldsymbol{\theta}^e(\boldsymbol{u}^e(z)-\frac{n}{2\pi}\boldsymbol{\Omega}^e+\boldsymbol{d}_e)}{\boldsymbol{\theta}^e(\boldsymbol{u}^e(z)+\boldsymbol{d}_e)} & -\frac{(\gamma^e(z)-(\gamma^e(z))^{-1})}{2\mathrm{i}}\frac{\boldsymbol{\theta}^e(-\boldsymbol{u}^e(z)-\frac{n}{2\pi}\boldsymbol{\Omega}^e+\boldsymbol{d}_e)}{\boldsymbol{\theta}^e(-\boldsymbol{u}^e(z)+\boldsymbol{d}_e)} \\ \frac{(\gamma^e(z)-(\gamma^e(z))^{-1})}{2\mathrm{i}}\frac{\boldsymbol{\theta}^e(\boldsymbol{u}^e(z)-\frac{n}{2\pi}\boldsymbol{\Omega}^e-\boldsymbol{d}_e)}{\boldsymbol{\theta}^e(\boldsymbol{u}^e(z)-\boldsymbol{d}_e)} & \frac{(\gamma^e(z)+(\gamma^e(z))^{-1})}{2}\frac{\boldsymbol{\theta}^e(-\boldsymbol{u}^e(z)-\frac{n}{2\pi}\boldsymbol{\Omega}^e-\boldsymbol{d}_e)}{\boldsymbol{\theta}^e(\boldsymbol{u}^e(z)+\boldsymbol{d}_e)} \end{pmatrix},$$

with $\gamma^e(z)$ *characterised completely in Lemma* 4.4, $\boldsymbol{\Omega}^e := (\Omega_1^e, \Omega_2^e, \dots, \Omega_N^e)^{\mathrm{T}}$ $(\in \mathbb{R}^N)$, *where* $\Omega_j^e = 4\pi\int_{b_j^e}^{a_{N+1}^e}\psi_V^e(s)\,\mathrm{d}s$, $j=1,\dots,N$, *and* $^{\mathrm{T}}$ *denotes transposition,* $\boldsymbol{d}_e \equiv -\sum_{j=1}^N\int_{a_j^e}^{z_j^{e,-}}\boldsymbol{\omega}^e$ $(=\sum_{j=1}^N\int_{a_j^e}^{z_j^{e,+}}\boldsymbol{\omega}^e)$, $\{z_j^{e,\pm}\}_{j=1}^N$ *are characterised completely in Lemma* 4.4, $\boldsymbol{\omega}^e$ *is the associated normalised basis of holomorphic one-forms of* $\mathcal{Y}_e$, $\boldsymbol{u}^e(z) := \int_{a_{N+1}^e}^z \boldsymbol{\omega}^e$ $(\in \mathrm{Jac}(\mathcal{Y}_e))$, *and* $\boldsymbol{u}_+^e(\infty) := \int_{a_{N+1}^e}^{\infty^+}\boldsymbol{\omega}^e$ *(*∞^+ *is the point at infinity in* $\mathbb{C}_+$*); furthermore, the solution is unique.*

Proof. Let $\overset{e}{m}{}^{\infty} \colon \mathbb{C} \setminus J_e^{\infty} \to \mathrm{SL}_2(\mathbb{C})$ solve the RHP formulated in Lemma 4.3, and define $\overset{e}{m}{}^{\infty}(z)$, in terms of $\overset{e}{\mathfrak{M}}{}^{\infty}(z)$, as in the Lemma. A straightforward calculation shows that $\overset{e}{\mathfrak{M}}{}^{\infty} \colon \mathbb{C} \setminus \mathbb{R} \to \mathrm{SL}_2(\mathbb{C})$ solves the following (normalised at infinity) 'twisted' RHP: (i) $\overset{e}{\mathfrak{M}}{}^{\infty}(z)$ is holomorphic for $z \in \mathbb{C} \setminus \widetilde{J}^e$, where $\widetilde{J}^e := (-\infty, b_0^e) \cup (a_{N+1}^e, +\infty) \cup (\cup_{j=1}^N (a_j^e, b_j^e))$; (ii) $\overset{e}{\mathfrak{M}}{}^{\infty}_{\pm}(z) := \lim_{\substack{z' \to z \\ z' \in \pm \,\mathrm{side\,of}\, \widetilde{J}^e}} \overset{e}{\mathfrak{M}}{}^{\infty}(z')$ satisfy the boundary condition $\overset{e}{\mathfrak{M}}{}^{\infty}_{+}(z) = \overset{e}{\mathfrak{M}}{}^{\infty}_{-}(z) \overset{e}{\mathcal{V}}{}^{\infty}(z)$, $z \in \widetilde{J}^e$, where

$$\overset{e}{\mathcal{V}}{}^{\infty}(z) := \begin{cases} \mathrm{I}, & z \in J_e, \\ -\mathrm{i}\sigma_2, & z \in (-\infty, b_0^e) \cup (a_{N+1}^e, +\infty), \\ -\mathrm{i}\sigma_2 \mathrm{e}^{-\mathrm{i}n\Omega_j^e \sigma_3}, & z \in (a_j^e, b_j^e), \quad j = 1, \dots, N, \end{cases} \tag{4.1}$$

with $\Omega_j^e = 4\pi \int_{b_j^e}^{a_{N+1}^e} \psi_V^e(s)\,\mathrm{d}s$, $j = 1, \dots, N$; (iii) $\overset{e}{\mathfrak{M}}{}^{\infty}(z) \underset{z \in \mathbb{C}_+}{=_{z \to \infty}} \mathrm{I} + \mathcal{O}(z^{-1})$ and $\overset{e}{\mathfrak{M}}{}^{\infty}(z) \underset{z \in \mathbb{C}_-}{=_{z \to \infty}} \mathrm{i}\sigma_2 + \mathcal{O}(z^{-1})$; and (iv) $\overset{e}{\mathfrak{M}}{}^{\infty}(z) \underset{z \in \mathbb{C} \setminus \widetilde{J}^e}{=_{z \to 0}} \mathcal{O}(1)$. The solution of this latter (twisted) RHP for $\overset{e}{\mathfrak{M}}{}^{\infty}(z)$ is constructed out of the solution of two, simpler RHPs: $(\mathcal{N}^e(z), -\mathrm{i}\sigma_2, \widetilde{J}^e)$ and $(\overset{e}{\mathfrak{m}}{}^{\infty}(z), \overset{e}{\mathcal{U}}{}^{\infty}(z), \widetilde{J}^e)$, where $\overset{e}{\mathcal{U}}{}^{\infty}(z)$ equals $\exp(\mathrm{i}n\Omega_j^e \sigma_3)\sigma_1$ for $z \in (a_j^e, b_j^e)$, $j = 1, \dots, N$, and equals I for $z \in (-\infty, b_0^e) \cup (a_{N+1}^e, +\infty)$. The RHP $(\mathcal{N}^e(z), -\mathrm{i}\sigma_2, \widetilde{J}^e)$ is solved explicitly by diagonalising the jump matrix, and thus reduced to two scalar RHPs [2] (see, also, [57, 59, 90]): the solution is

$$\mathcal{N}^e(z) = \begin{pmatrix} \frac{1}{2}(\gamma^e(z) + (\gamma^e(z))^{-1}) & -\frac{1}{2\mathrm{i}}(\gamma^e(z) - (\gamma^e(z))^{-1}) \\ \frac{1}{2\mathrm{i}}(\gamma^e(z) - (\gamma^e(z))^{-1}) & \frac{1}{2}(\gamma^e(z) + (\gamma^e(z))^{-1}) \end{pmatrix},$$

where $\gamma^e \colon \mathbb{C} \setminus \widetilde{J}^e \to \mathbb{C}$ is characterised completely in Lemma 4.4; furthermore, $\mathcal{N}^e(z)$ is piecewise holomorphic for $z \in \mathbb{C} \setminus \widetilde{J}^e$, and $\mathcal{N}^e(z) \underset{z \in \mathbb{C}_+}{=_{z \to \infty}} \mathrm{I} + \mathcal{O}(z^{-1})$ and $\mathcal{N}^e(z) \underset{z \in \mathbb{C}_-}{=_{z \to \infty}} \mathrm{i}\sigma_2 + \mathcal{O}(z^{-1})$[11].

Consider, now, the functions $\boldsymbol{\theta}^e(\boldsymbol{u}^e(z) \pm \boldsymbol{d}_e)$, where $\boldsymbol{u}^e(z) \colon z \to \mathrm{Jac}(\mathcal{Y}_e)$, $z \mapsto \boldsymbol{u}^e(z) := \int_{a_{N+1}^e}^{z} \boldsymbol{\omega}^e$, with $\boldsymbol{\omega}^e$ the associated normalised basis of holomorphic one-forms of $\mathcal{Y}_e$, $\boldsymbol{d}_e \equiv -\sum_{j=1}^N \int_{a_j^e}^{z_j^{e,-}} \boldsymbol{\omega}^e = \sum_{j=1}^N \int_{a_j^e}^{z_j^{e,+}} \boldsymbol{\omega}^e$, where $\equiv$ denotes equivalence modulo the period lattice, and $\{z_j^{e,\pm}\}_{j=1}^N$ are characterised completely in Lemma 4.4. From the general theory of theta functions on Riemann surfaces (see, for example, [87, 88]), $\boldsymbol{\theta}^e(\boldsymbol{u}^e(z) + \boldsymbol{d}_e)$, for $z \in \mathcal{Y}_e := \{(y, z);\ y^2 = \prod_{k=1}^{N+1}(z - b_{k-1}^e)(z - a_k^e)\}$, is either identically zero on $\mathcal{Y}_e$ or has precisely N (simple) zeros (the generic case). In this case, since the divisors $\prod_{j=1}^N z_j^{e,-}$ and $\prod_{j=1}^N z_j^{e,+}$ are non-special, one uses Lemma 3.27 of [57] (see, also, Lemma 4.2 of [58]) and the representation [88] $\boldsymbol{K}_e = \sum_{j=1}^N \int_{a_j^e}^{a_{N+1}^e} \boldsymbol{\omega}^e$, for the 'even' vector of Riemann constants, with $2\boldsymbol{K}_e = 0$ and $s\boldsymbol{K}_e \neq 0$, $0 < s < 2$, to arrive at

$$\begin{aligned} \boldsymbol{\theta}^e(\boldsymbol{u}^e(z) + \boldsymbol{d}_e) &= \boldsymbol{\theta}^e\left(\boldsymbol{u}^e(z) - \sum_{j=1}^N \int_{a_j^e}^{z_j^{e,-}} \boldsymbol{\omega}^e\right) = \boldsymbol{\theta}^e\left(\int_{a_{N+1}^e}^{z} \boldsymbol{\omega}^e - \boldsymbol{K}_e - \sum_{j=1}^N \int_{a_{N+1}^e}^{z_j^{e,-}} \boldsymbol{\omega}^e\right) = 0 \\ &\Leftrightarrow z \in \left\{z_1^{e,-}, z_2^{e,-}, \dots, z_N^{e,-}\right\}, \\ \boldsymbol{\theta}^e(\boldsymbol{u}^e(z) - \boldsymbol{d}_e) &= \boldsymbol{\theta}^e\left(\boldsymbol{u}^e(z) - \sum_{j=1}^N \int_{a_j^e}^{z_j^{e,+}} \boldsymbol{\omega}^e\right) = \boldsymbol{\theta}^e\left(\int_{a_{N+1}^e}^{z} \boldsymbol{\omega}^e - \boldsymbol{K}_e - \sum_{j=1}^N \int_{a_{N+1}^e}^{z_j^{e,+}} \boldsymbol{\omega}^e\right) = 0 \\ &\Leftrightarrow z \in \left\{z_1^{e,+}, z_2^{e,+}, \dots, z_N^{e,+}\right\}. \end{aligned}$$

---

[11] Note that, strictly speaking, $\mathcal{N}^e(z)$, as given above, does not solve the RHP $(\mathcal{N}^e(z), -\mathrm{i}\sigma_2, \widetilde{J}^e)$ in the sense defined heretofore, as $\mathcal{N}^e \upharpoonright_{\mathbb{C}_\pm}$ can not be extended continuously to $\overline{\mathbb{C}}_\pm$; however, $\mathcal{N}^e(\cdot \pm \mathrm{i}\varepsilon)$ converge in $\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C}),\mathrm{loc}}(\mathbb{R})$ as $\varepsilon \downarrow 0$ to $\mathrm{SL}_2(\mathbb{C})$-valued functions $\mathcal{N}^e(z)$ in $\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{J}^e)$ that satisfy $\mathcal{N}^e_+(z) = \mathcal{N}^e_-(z)(-\mathrm{i}\sigma_2)$ a.e. on $\widetilde{J}^e$: one then shows that $\mathcal{N}^e(z)$ is the unique solution of the RHP $(\mathcal{N}^e(z), -\mathrm{i}\sigma_2, \widetilde{J}^e)$, where the latter boundary/jump condition is interpreted in the $\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C}),\mathrm{loc}}$ sense.

Following Lemma 3.21 of [57], set

$$\overset{e}{\mathfrak{m}}^{\infty}(z):=\begin{pmatrix} \frac{\boldsymbol{\theta}^e(\boldsymbol{u}^e(z)-\frac{n}{2\pi}\boldsymbol{\Omega}^e+\boldsymbol{d}_e)}{\boldsymbol{\theta}^e(\boldsymbol{u}^e(z)+\boldsymbol{d}_e)} & \frac{\boldsymbol{\theta}^e(-\boldsymbol{u}^e(z)-\frac{n}{2\pi}\boldsymbol{\Omega}^e+\boldsymbol{d}_e)}{\boldsymbol{\theta}^e(-\boldsymbol{u}^e(z)+\boldsymbol{d}_e)} \\ \frac{\boldsymbol{\theta}^e(\boldsymbol{u}^e(z)-\frac{n}{2\pi}\boldsymbol{\Omega}^e-\boldsymbol{d}_e)}{\boldsymbol{\theta}^e(\boldsymbol{u}^e(z)-\boldsymbol{d}_e)} & \frac{\boldsymbol{\theta}^e(-\boldsymbol{u}^e(z)-\frac{n}{2\pi}\boldsymbol{\Omega}^e-\boldsymbol{d}_e)}{\boldsymbol{\theta}^e(\boldsymbol{u}^e(z)+\boldsymbol{d}_e)} \end{pmatrix},$$

where $\boldsymbol{\Omega}^e:=(\Omega_1^e,\Omega_2^e,\dots,\Omega_N^e)^{\mathrm{T}}$ $(\in\mathbb{R}^N)$, with Ω_j^e, $j=1,\dots,N$, given above, and $^{\mathrm{T}}$ denoting transposition. Using Lemma 3.18 of [57] (or, equivalently, Equations (4.65) and (4.66) of [58]), that is, for $z\in(a_j^e,b_j^e)$, $j=1,\dots,N$, $\boldsymbol{u}_+^e(z)+\boldsymbol{u}_-^e(z)\equiv-\tau_j^e$ $(:=-\tau^e e_j)$, $j=1,\dots,N$, with $\tau^e:=(\tau^e)_{i,j=1,\dots,N}:=(\oint_{\boldsymbol{\beta}_j^e}\omega_i^e)_{i,j=1,\dots,N}$ (the associated matrix of Riemann periods), and, for $z\in(-\infty,b_0^e)\cup(a_{N+1}^e,+\infty)$, $\boldsymbol{u}_+^e(z)+\boldsymbol{u}_-^e(z)\equiv 0$, where $\boldsymbol{u}_\pm^e(z):=\int_{a_{N+1}^e}^{z^\pm}\boldsymbol{\omega}^e$, with $z^\pm\in(a_j^e,b_j^e)^\pm$, $j=1,\dots,N$, and the evenness and (quasi-) periodicity properties of $\boldsymbol{\theta}^e(z)$, one shows that, for $z\in(a_j^e,b_j^e)$, $j=1,\dots,N$,

$$\frac{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(z)-\frac{n}{2\pi}\boldsymbol{\Omega}^e+\boldsymbol{d}_e)}{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(z)+\boldsymbol{d}_e)}=\mathrm{e}^{-\mathrm{i}n\Omega_j^e}\frac{\boldsymbol{\theta}^e(-\boldsymbol{u}_-^e(z)-\frac{n}{2\pi}\boldsymbol{\Omega}^e+\boldsymbol{d}_e)}{\boldsymbol{\theta}^e(-\boldsymbol{u}_-^e(z)+\boldsymbol{d}_e)},$$
$$\frac{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(z)-\frac{n}{2\pi}\boldsymbol{\Omega}^e-\boldsymbol{d}_e)}{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(z)-\boldsymbol{d}_e)}=\mathrm{e}^{-\mathrm{i}n\Omega_j^e}\frac{\boldsymbol{\theta}^e(-\boldsymbol{u}_-^e(z)-\frac{n}{2\pi}\boldsymbol{\Omega}^e-\boldsymbol{d}_e)}{\boldsymbol{\theta}^e(\boldsymbol{u}_-^e(z)+\boldsymbol{d}_e)},$$
$$\frac{\boldsymbol{\theta}^e(-\boldsymbol{u}_+^e(z)-\frac{n}{2\pi}\boldsymbol{\Omega}^e+\boldsymbol{d}_e)}{\boldsymbol{\theta}^e(-\boldsymbol{u}_+^e(z)+\boldsymbol{d}_e)}=\mathrm{e}^{\mathrm{i}n\Omega_j^e}\frac{\boldsymbol{\theta}^e(\boldsymbol{u}_-^e(z)-\frac{n}{2\pi}\boldsymbol{\Omega}^e+\boldsymbol{d}_e)}{\boldsymbol{\theta}^e(\boldsymbol{u}_-^e(z)+\boldsymbol{d}_e)},$$
$$\frac{\boldsymbol{\theta}^e(-\boldsymbol{u}_+^e(z)-\frac{n}{2\pi}\boldsymbol{\Omega}^e-\boldsymbol{d}_e)}{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(z)+\boldsymbol{d}_e)}=\mathrm{e}^{\mathrm{i}n\Omega_j^e}\frac{\boldsymbol{\theta}^e(\boldsymbol{u}_-^e(z)-\frac{n}{2\pi}\boldsymbol{\Omega}^e-\boldsymbol{d}_e)}{\boldsymbol{\theta}^e(\boldsymbol{u}_-^e(z)-\boldsymbol{d}_e)},$$

and, for $z\in(-\infty,b_0^e)\cup(a_{N+1}^e,+\infty)$, one obtains the same relations as above but with the changes $\exp(\mp\mathrm{i}n\Omega_j^e)\to 1$. Set, as in Proposition 3.31 of [57],

$$\overset{e}{\mathcal{Q}}^{\infty}(z):=\begin{pmatrix} (\mathcal{N}^e(z))_{11}(\overset{e}{\mathfrak{m}}^{\infty}(z))_{11} & (\mathcal{N}^e(z))_{12}(\overset{e}{\mathfrak{m}}^{\infty}(z))_{12} \\ (\mathcal{N}^e(z))_{21}(\overset{e}{\mathfrak{m}}^{\infty}(z))_{21} & (\mathcal{N}^e(z))_{22}(\overset{e}{\mathfrak{m}}^{\infty}(z))_{22} \end{pmatrix},$$

where $(*)_{ij}$, $i,j=1,2$, denotes the $(i\,j)$-element of $(*)$. Recalling that $\mathcal{N}^e\colon\mathbb{C}\setminus\widetilde{J}^e\to\mathrm{SL}_2(\mathbb{C})$ solves the RHP $(\mathcal{N}^e(z),-\mathrm{i}\sigma_2,\widetilde{J}^e)$, using the above theta-functional relations and the large-z asymptotic expansion of $\boldsymbol{u}^e(z)$ (see Section 5, the proof of Proposition 5.3), one shows that $\overset{e}{\mathcal{Q}}^{\infty}(z)$ solves the following RHP: (i) $\overset{e}{\mathcal{Q}}^{\infty}(z)$ is holomorphic for $z\in\mathbb{C}\setminus\widetilde{J}^e$; (ii) $\overset{e}{\mathcal{Q}}^{\infty}_{\pm}(z):=\lim_{\substack{z'\to z\\ z'\in\pm\text{ side of }\widetilde{J}^e}}\overset{e}{\mathcal{Q}}^{\infty}(z')$ satisfy the boundary condition $\overset{e}{\mathcal{Q}}^{\infty}_{+}(z)=\overset{e}{\mathcal{Q}}^{\infty}_{-}(z)\overset{e}{\mathcal{V}}^{\infty}(z)$, $z\in\widetilde{J}^e$, where $\overset{e}{\mathcal{V}}^{\infty}(z)$ is defined in Equation (4.1); (iii)

$$\overset{e}{\mathcal{Q}}^{\infty}(z)\underset{\substack{z\to\infty\\ z\in\mathbb{C}_+}}{=}\begin{pmatrix} \frac{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\infty)-\frac{n}{2\pi}\boldsymbol{\Omega}^e+\boldsymbol{d}_e)}{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\infty)+\boldsymbol{d}_e)} & 0 \\ 0 & \frac{\boldsymbol{\theta}^e(-\boldsymbol{u}_+^e(\infty)-\frac{n}{2\pi}\boldsymbol{\Omega}^e-\boldsymbol{d}_e)}{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\infty)+\boldsymbol{d}_e)} \end{pmatrix}+\mathcal{O}(z^{-1}),$$
$$\overset{e}{\mathcal{Q}}^{\infty}(z)\underset{\substack{z\to\infty\\ z\in\mathbb{C}_-}}{=}\begin{pmatrix} 0 & \frac{\boldsymbol{\theta}^e(-\boldsymbol{u}_-^e(\infty)-\frac{n}{2\pi}\boldsymbol{\Omega}^e+\boldsymbol{d}_e)}{\boldsymbol{\theta}^e(-\boldsymbol{u}_-^e(\infty)+\boldsymbol{d}_e)} \\ -\frac{\boldsymbol{\theta}^e(\boldsymbol{u}_-^e(\infty)-\frac{n}{2\pi}\boldsymbol{\Omega}^e-\boldsymbol{d}_e)}{\boldsymbol{\theta}^e(\boldsymbol{u}_-^e(\infty)-\boldsymbol{d}_e)} & 0 \end{pmatrix}+\mathcal{O}(z^{-1}),$$

where $\boldsymbol{u}_\pm^e(\infty):=\int_{a_{N+1}^e}^{\infty^\pm}\boldsymbol{\omega}^e$ ($\infty^\pm$, respectively, are the points at infinity in $\mathbb{C}_\pm$); and (iv) $\overset{e}{\mathcal{Q}}^{\infty}(z)\underset{\substack{z\to 0\\ z\in\mathbb{C}\setminus\widetilde{J}^e}}{=}\mathcal{O}(1)$. Now, using the fact that $\boldsymbol{u}_-^e(\infty)=\int_{a_{N+1}^e}^{\infty^-}\boldsymbol{\omega}^e=-\int_{a_{N+1}^e}^{\infty^+}\boldsymbol{\omega}^e=-\boldsymbol{u}_+^e(\infty)$, upon multiplying $\overset{e}{\mathcal{Q}}^{\infty}(z)$ on the left by

$$\mathrm{diag}\left(\frac{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\infty)+\boldsymbol{d}_e)}{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\infty)-\frac{n}{2\pi}\boldsymbol{\Omega}^e+\boldsymbol{d}_e)},\frac{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\infty)+\boldsymbol{d}_e)}{\boldsymbol{\theta}^e(-\boldsymbol{u}_+^e(\infty)-\frac{n}{2\pi}\boldsymbol{\Omega}^e-\boldsymbol{d}_e)}\right)=:\overset{e}{\mathfrak{c}}^{\infty},$$

that is, $\overset{e}{\mathcal{Q}}^{\infty}(z)\to\overset{e}{\mathfrak{c}}^{\infty}\overset{e}{\mathcal{Q}}^{\infty}(z)=:\mathcal{M}_e^\infty(z)$, one easily shows that $\mathcal{M}_e^\infty\colon\mathbb{C}\setminus\widetilde{J}^e\to\mathrm{SL}_2(\mathbb{C})$ solves the RHP $(\mathcal{M}_e^\infty(z),\overset{e}{\mathcal{V}}^{\infty}(z),\widetilde{J}^e)$. Using, finally, the formula $\overset{e}{m}^{\infty}(z)=\begin{cases}\overset{e}{\mathfrak{M}}^{\infty}(z), & z\in\mathbb{C}_+,\\ -\mathrm{i}\overset{e}{\mathfrak{M}}^{\infty}(z)\sigma_2, & z\in\mathbb{C}_-,\end{cases}$ one shows that $\overset{e}{m}^{\infty}\colon\mathbb{C}\setminus$

$J_e^{\infty} \to \mathrm{SL}_2(\mathbb{C})$ solves the model RHP formulated in Lemma 4.3. One notes from the formula for $\overset{e}{\mathfrak{M}}{}^{\infty}(z)$ stated in the Lemma that it is well defined for $\mathbb{C} \setminus \mathbb{R}$; in particular, it is single valued and analytic (see below) for $z \in \mathbb{C} \setminus \widetilde{J}^e$ (independently of the path in $\mathbb{C} \setminus \widetilde{J}^e$ chosen to evaluate $\boldsymbol{u}^e(z) = \int_{a_{N+1}^e}^{z} \boldsymbol{\omega}^e$). Furthermore (cf. Lemma 4.4 and the analysis above), since $\{z' \in \mathbb{C};\, \boldsymbol{\theta}^e(\boldsymbol{u}^e(z') \pm \boldsymbol{d}_e) = 0\} = \{z_j^{e,\mp}\}_{j=1}^{N} = \{z' \in \mathbb{C};\, (\gamma^e(z) \pm (\gamma^e(z))^{-1})|_{z=z'} = 0\}$, one notes that the (simple) poles of $(\overset{e}{\mathfrak{m}}{}^{\infty}(z))_{11}$ and $(\overset{e}{\mathfrak{m}}{}^{\infty}(z))_{22}$ (resp., $(\overset{e}{\mathfrak{m}}{}^{\infty}(z))_{12}$ and $(\overset{e}{\mathfrak{m}}{}^{\infty}(z))_{21}$), that is, $\{z' \in \mathbb{C};\, \boldsymbol{\theta}^e(\boldsymbol{u}^e(z') + \boldsymbol{d}_e) = 0\}$ (resp., $\{z' \in \mathbb{C};\, \boldsymbol{\theta}^e(\boldsymbol{u}^e(z') - \boldsymbol{d}_e) = 0\}$), are exactly cancelled by the (simple) zeros of $\gamma^e(z) + (\gamma^e(z))^{-1}$ (resp., $\gamma^e(z) - (\gamma^e(z))^{-1}$); thus, $\overset{e}{\mathfrak{M}}{}^{\infty}(z)$ has only $\frac{1}{4}$-root singularities at the end-points of the support of the 'even' equilibrium measure, $\{b_{j-1}^e, a_j^e\}_{j=1}^{N+1}$. (This shows that $\overset{e}{\mathfrak{M}}{}^{\infty}(z)$ obtains its boundary values, $\overset{e}{\mathfrak{M}}{}^{\infty}_{\pm}(z) := \lim_{\varepsilon \downarrow 0} \overset{e}{\mathfrak{M}}{}^{\infty}(z \pm \mathrm{i}\varepsilon)$, in the $\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\mathbb{R})$ sense.) From the definition of $\overset{e}{m}{}^{\infty}(z)$ in terms of $\overset{e}{\mathfrak{M}}{}^{\infty}(z)$ given in the Lemma, the explicit formula for $\overset{e}{\mathfrak{M}}{}^{\infty}(z)$, and recalling that $\overset{e}{m}{}^{\infty}(z)$ solves the model RHP formulated in Lemma 4.3, one learns that, as $\det(\overset{e}{v}{}^{\infty}(z)) = 1$, $\det(\overset{e}{m}{}^{\infty}_{+}(z)) = \det(\overset{e}{m}{}^{\infty}_{-}(z))$, that is, $\det(\overset{e}{m}{}^{\infty}(z))$ has no 'jumps', whence $\det(\overset{e}{m}{}^{\infty}(z))$ has, at worst, (isolated) $\frac{1}{2}$-root singularities at $\{b_{j-1}^e, a_j^e\}_{j=1}^{N+1}$, which are removable, which implies that $\det(\overset{e}{m}{}^{\infty}(z))$ is entire and bounded; hence, via a generalisation of Liouville's Theorem, and the asymptotic relation $\det(\overset{e}{m}{}^{\infty}(z)) =_{\substack{z \to \infty \\ z \in \mathbb{C}\setminus\mathbb{R}}} 1 + \mathcal{O}(z^{-1})$, one arrives at $\det(\overset{e}{m}{}^{\infty}(z)) = 1 \Rightarrow \overset{e}{m}{}^{\infty}(z) \in \mathrm{SL}_2(\mathbb{C})$. Also, from the definition of $\overset{e}{m}{}^{\infty}(z)$ in terms of $\overset{e}{\mathfrak{M}}{}^{\infty}(z)$ and the explicit formula for $\overset{e}{\mathfrak{M}}{}^{\infty}(z)$, it follows that both $\overset{e}{m}{}^{\infty}(z)$ and $(\overset{e}{m}{}^{\infty}(z))^{-1}$ are uniformly bounded as functions of n (as $n \to \infty$) for z in compact subsets away from $\{b_{j-1}^e, a_j^e\}_{j=1}^{N+1}$.

Let $\mathcal{S}_e^{\infty} \colon \mathbb{C} \setminus \mathbb{R} \to \mathrm{SL}_2(\mathbb{C})$ be another solution of the RHP $(\overset{e}{\mathfrak{M}}{}^{\infty}(z), \overset{e}{\mathcal{V}}{}^{\infty}(z), \mathbb{R})$ formulated at the beginning of the proof, and set, as in the *Remark on Proposition* 3.43 of [57], $\Delta^e(z) := \overset{e}{\mathfrak{M}}{}^{\infty}(z) - \mathcal{S}_e^{\infty}(z)$, whence $\Delta^e =_{\substack{z \to \infty \\ z \in \mathbb{C}\setminus\mathbb{R}}} \mathcal{O}(z^{-1})$; thus, by Cauchy's Theorem, $\int_{\mathrm{C}_\epsilon^e} \Delta^e(s)(\Delta^e(\overline{s}))^{\dagger}\,\mathrm{d}s = 0$, where † denotes the Hermitean adjoint, and C_ϵ^e is the (closed and simple) counter-clockwise-oriented contour $\mathrm{C}_\epsilon^e := \mathrm{C}_\epsilon^{e,\mathbb{R}} \cup \mathrm{C}_\epsilon^{e,\frown}$, where $\mathrm{C}_\epsilon^{e,\mathbb{R}} := \{x + \mathrm{i}\epsilon;\, -\epsilon^{-1} \leqslant x \leqslant \epsilon^{-1}\}$ and $\mathrm{C}_\epsilon^{e,\frown} := \{\epsilon^{-1}\mathrm{e}^{\mathrm{i}\theta};\, \theta \in [\delta(\epsilon), \pi - \delta(\epsilon)],\, \delta(\epsilon) := \tan^{-1}(\epsilon^2)\}$, with ϵ some arbitrarily fixed, sufficiently small positive real number, and the principal branch of $\tan^{-1}(\cdot)$ is taken. Writing $0 = \int_{\mathrm{C}_\epsilon^e} \Delta^e(s)(\Delta^e(\overline{s}))^{\dagger}\,\mathrm{d}s = (\int_{\mathrm{C}_\epsilon^{e,\mathbb{R}}} + \int_{\mathrm{C}_\epsilon^{e,\frown}})\Delta^e(s)(\Delta^e(\overline{s}))^{\dagger}\mathrm{d}s$, letting $\epsilon \downarrow 0$, in which case, since $\Delta^e(z)(\Delta^e(\overline{z}))^{\dagger} =_{\substack{z \to \infty \\ z \in \mathbb{C}\setminus\mathbb{R}}} \mathcal{O}(z^{-2})$, an application of Jordan's Lemma gives $\int_{\mathrm{C}_\epsilon^{e,\frown}} \Delta^e(s)(\Delta^e(\overline{s}))^{\dagger}\,\mathrm{d}s =_{\epsilon \downarrow 0} 0$, one gets that

$$
\begin{aligned}
0 = \int_{-\infty}^{+\infty} \Delta_+^e(s)(\Delta_-^e(s))^{\dagger}\,\mathrm{d}s &= \int_{-\infty}^{b_0^e} \Delta_-^e(s)(-\mathrm{i}\sigma_2)(\Delta_-^e(s))^{\dagger}\,\mathrm{d}s + \int_{a_{N+1}^e}^{+\infty} \Delta_-^e(s)(-\mathrm{i}\sigma_2)(\Delta_-^e(s))^{\dagger}\,\mathrm{d}s \\
&+ \sum_{j=1}^{N} \int_{a_j^e}^{b_j^e} \Delta_-^e(s)(-\mathrm{i}\sigma_2 \mathrm{e}^{-\mathrm{i}n\Omega_j^e\sigma_3})(\Delta_-^e(s))^{\dagger}\,\mathrm{d}s + \int_{J_e} \Delta^e(s)(\Delta^e(s))^{\dagger}\,\mathrm{d}s :
\end{aligned}
$$

adding the above to its Hermitean adjoint, that is,

$$
\begin{aligned}
0 = &\int_{-\infty}^{b_0^e} \Delta_-^e(s)(\mathrm{i}\sigma_2)(\Delta_-^e(s))^{\dagger}\,\mathrm{d}s + \int_{a_{N+1}^e}^{+\infty} \Delta_-^e(s)(\mathrm{i}\sigma_2)(\Delta_-^e(s))^{\dagger}\,\mathrm{d}s \\
&+ \sum_{j=1}^{N} \int_{a_j^e}^{b_j^e} \Delta_-^e(s)(\mathrm{i}\mathrm{e}^{\mathrm{i}n\Omega_j^e\sigma_3}\sigma_2)(\Delta_-^e(s))^{\dagger}\,\mathrm{d}s + \int_{J_e} \Delta^e(s)(\Delta^e(s))^{\dagger}\,\mathrm{d}s,
\end{aligned}
$$

one arrives at $2\int_{J_e} \Delta^e(s)(\Delta^e(s))^{\dagger}\,\mathrm{d}s = 0$; thus, $\Delta^e(z) = 0$, $z \in J_e$, which implies that $\overset{e}{\mathfrak{M}}{}^{\infty}(z) = \mathcal{S}_e^{\infty}(z)$ for all z. □

In order to prove that there is a solution of the (full) RHP $(\overset{e}{\mathcal{M}}{}^{\sharp}(z), \overset{e}{v}{}^{\sharp}(z), \Sigma_e^{\sharp})$, formulated in Lemma 4.2, close to the parametrix, one needs to know that the parametrix is *uniformly* bounded: more precisely, by (certain) general theorems (see, for example, [98]), one needs to know that $\overset{e}{v}{}^{\sharp}(z) \to \overset{e}{v}{}^{\infty}(z)$

as $n \to \infty$ uniformly for $z \in \Sigma_e^\sharp$ in the $\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Sigma_e^\sharp) \cap \mathcal{L}^\infty_{\mathrm{M}_2(\mathbb{C})}(\Sigma_e^\sharp)$ sense, that is, uniformly,

$$\lim_{n\to\infty} \|\overset{e}{v}{}^\sharp(\cdot) - \overset{e}{v}{}^\infty(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Sigma_e^\sharp)\cap\mathcal{L}^\infty_{\mathrm{M}_2(\mathbb{C})}(\Sigma_e^\sharp)} := \lim_{n\to\infty} \sum_{p\in\{2,\infty\}} \|\overset{e}{v}{}^\sharp(\cdot) - \overset{e}{v}{}^\infty(\cdot)\|_{\mathcal{L}^p_{\mathrm{M}_2(\mathbb{C})}(\Sigma_e^\sharp)} = 0;$$

however, notwithstanding the fact that $\widetilde{V} \colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is regular ($h_V^e(b_{j-1}^e), h_V^e(a_j^e) \neq 0$, $j = 1, \dots, N+1$), since the strict inequalities $g_+^e(z) + g_-^e(z) - \widetilde{V}(z) - \ell_e + 2Q_e < 0$, $z \in (-\infty, b_0^e) \cup (a_{N+1}^e, +\infty) \cup (\cup_{j=1}^N (a_j^e, b_j^e))$, and $\pm\mathrm{Re}(\mathrm{i}\int_z^{a_{N+1}^e} \psi_V^e(s)\,\mathrm{d}s) > 0$, $z \in \mathbb{C}_\pm \cap (\cup_{j=1}^{N+1} \mathbb{U}_j^e)$, fail at the end-points of the support of the 'even' equilibrium measure, this implies that $\overset{e}{v}{}^\sharp(z) \to \overset{e}{v}{}^\infty(z)$ as $n \to \infty$ *pointwise*, but not uniformly, for $z \in \Sigma_e^\sharp$, whence, one can not conclude that $\overset{e}{\mathcal{M}}{}^\sharp(z) \to \overset{e}{m}{}^\infty(z)$ as $n \to \infty$ uniformly for $z \in \Sigma_e^\sharp$. The resolution of this lack of uniformity at the end-points of the support of the 'even' equilibrium measure constitutes, therefore, the essential analytical obstacle remaining for the analysis of the RHP $(\overset{e}{\mathcal{M}}{}^\sharp(z), \overset{e}{v}{}^\sharp(z), \Sigma_e^\sharp)$, and a substantial part of the following analysis is devoted to overcoming this problem.

The key necessary to remedy (and control) the above-mentioned analytical difficulty is to construct parametrices for the solution of the RHP $(\overset{e}{\mathcal{M}}{}^\sharp(z), \overset{e}{v}{}^\sharp(z), \Sigma_e^\sharp)$ in 'small' neighbourhoods (open discs) about $\{b_{j-1}^e, a_j^e\}_{j=1}^{N+1}$ (where the convergence of $\overset{e}{v}{}^\sharp(z)$ to $\overset{e}{v}{}^\infty(z)$ as $n \to \infty$ is not uniform) in such a way that, on the boundary of these open neighbourhoods, the parametrices 'match' with the solution of the model RHP, $\overset{e}{m}{}^\infty(z)$, up to $o(1)$ (in fact, $\mathcal{O}(n^{-1})$) as $n \to \infty$; furthermore, in the generic framework considered in this work, namely, $\widetilde{V} \colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is regular, in which case the (density of the) 'even' equilibrium measure behaves as a square root at the end-points of $\mathrm{supp}(\mu_V^e)$, that is, $\psi_V^e(s) =_{s\downarrow b_{j-1}^e} \mathcal{O}((s-b_{j-1}^e)^{1/2})$ and $\psi_V^e(s) =_{s\uparrow a_j^e} \mathcal{O}((a_j^e - s)^{1/2})$, $j = 1, \dots, N+1$, it is well known [3, 59, 90, 99] that the parametrices can be expressed in terms of Airy functions. (The general method used to construct such parametrices is via a 'Vanishing Lemma' [100].) More precisely, one surrounds the end-points of the support of the 'even' equilibrium measure, $\{b_{j-1}^e, a_j^e\}_{j=1}^{N+1}$, by 'small', mutually disjoint open discs,

$$\mathbb{D}_\epsilon(b_{j-1}^e) := \left\{z \in \mathbb{C};\ |z - b_{j-1}^e| < \epsilon_j^b\right\} \qquad \text{and} \qquad \mathbb{D}_\epsilon(a_j^e) := \left\{z \in \mathbb{C};\ |z - a_j^e| < \epsilon_j^a\right\}, \quad j = 1, \dots, N+1,$$

where $\epsilon_j^b, \epsilon_j^a$ are arbitrarily fixed, sufficiently small positive real numbers chosen so that $\mathbb{D}_\epsilon(b_{i-1}^e) \cap \mathbb{D}_\epsilon(a_j^e) = \emptyset$, $i, j = 1, \dots, N+1$, and defines $S_p^e(z)$, the parametrix for $\overset{e}{\mathcal{M}}{}^\sharp(z)$, by $\overset{e}{m}{}^\infty(z)$ for $z \in \mathbb{C} \setminus (\cup_{j=1}^{N+1} (\mathbb{D}_\epsilon(b_{j-1}^e) \cup \mathbb{D}_\epsilon(a_j^e)))$, and by $m_p^e(z)$ for $z \in \cup_{j=1}^{N+1} (\mathbb{D}_\epsilon(b_{j-1}^e) \cup \mathbb{D}_\epsilon(a_j^e))$, and solves the local RHPs for $m_p^e(z)$ on $\cup_{j=1}^{N+1} (\mathbb{D}_\epsilon(b_{j-1}^e) \cup \mathbb{D}_\epsilon(a_j^e))$ in such a way ('optimal' in the nomenclature of [59]) that $m_p^e(z) \approx_{n\to\infty} \overset{e}{m}{}^\infty(z)$ (to $\mathcal{O}(n^{-1})$) for $z \in \cup_{j=1}^{N+1} (\partial\mathbb{D}_\epsilon(b_{j-1}^e) \cup \partial\mathbb{D}_\epsilon(a_j^e))$, whence $\mathcal{R}^e(z) := \overset{e}{\mathcal{M}}{}^\sharp(z)(S_p^e(z))^{-1} \colon \mathbb{C} \setminus \widetilde{\Sigma}_e^\sharp \to \mathrm{SL}_2(\mathbb{C})$, where $\widetilde{\Sigma}_e^\sharp := \Sigma_e^\sharp \cup (\cup_{j=1}^{N+1} (\partial\mathbb{D}_\epsilon(b_{j-1}^e) \cup \partial\mathbb{D}_\epsilon(a_j^e)))$, solves the RHP $(\mathcal{R}^e(z), v_{\mathcal{R}}^e(z), \widetilde{\Sigma}_e^\sharp)$, with $\|v_{\mathcal{R}}^e(\cdot) - \mathrm{I}\|_{\cap_{p\in\{2,\infty\}} \mathcal{L}^p_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}_e^\sharp)} =_{n\to\infty} \mathcal{O}(n^{-1})$ uniformly; in particular, the error term, which is $\mathcal{O}(n^{-1})$ as $n \to \infty$, is uniform in $\cap_{p\in\{1,2,\infty\}} \mathcal{L}^p_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}_e^\sharp)$. By general Riemann-Hilbert techniques (see, for example, [98]), $\mathcal{R}^e(z)$ (and thus $\overset{e}{\mathcal{M}}{}^\sharp(z)$ via the relation $\overset{e}{\mathcal{M}}{}^\sharp(z) = \mathcal{R}^e(z) S_p^e(z)$) can be computed to any order of n^{-1} via a Neumann series expansion (of the corresponding resolvent kernel). In fact, at the very core of the above-mentioned discussion, and the analysis that follows, is the following Corollary (see, for example, [90], Corollary 7.108):

Corollary 4.1 (Deift [90]). *For an oriented contour $\Sigma \subset \mathbb{C}$, let $m^\infty \colon \mathbb{C} \setminus \Sigma \to \mathrm{SL}_2(\mathbb{C})$ and $m^{(n)} \colon \mathbb{C} \setminus \Sigma \to \mathrm{SL}_2(\mathbb{C})$, $n \in \mathbb{N}$, respectively, solve the following, equivalent* RHPs, *$(m^\infty(z), v^\infty(z), \Sigma)$ and $(m^{(n)}(z), v^{(n)}(z), \Sigma)$, where*

$$v^\infty \colon \Sigma \to \mathrm{GL}_2(\mathbb{C}),\ z \mapsto (\mathrm{I} - w_-^\infty(z))^{-1}(\mathrm{I} + w_+^\infty(z))$$

and

$$v^{(n)} \colon \Sigma \to \mathrm{GL}_2(\mathbb{C}),\ z \mapsto \left(\mathrm{I} - w_-^{(n)}(z)\right)^{-1}\left(\mathrm{I} + w_+^{(n)}(z)\right),$$

and suppose that $(\mathbf{id} - C_{w^\infty}^\infty)^{-1}$ exists, where

$$\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Sigma) \ni f \mapsto C_{w^\infty}^\infty f := C_+^\infty(f w_-^\infty) + C_-^\infty(f w_+^\infty),$$

with

$$C^{\infty}_{\pm}\colon \mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Sigma)\to\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Sigma),\ f\mapsto(C^{\infty}_{\pm}f)(z):=\lim_{\substack{z'\to z\\ z'\,\in\,\pm\,\text{side of}\,\Sigma}}\int_{\Sigma}\frac{f(s)}{s-z'}\frac{\mathrm{d}s}{2\pi\mathrm{i}},$$

and $\|w^{(n)}_l(\cdot)-w^{\infty}_l(\cdot)\|_{\cap_{p\in[2,\infty]}\mathcal{L}^p_{\mathrm{M}_2(\mathbb{C})}(\Sigma)}\to 0$ *as* $n\to\infty$, $l=\pm1$. *Then,* $\exists\ N^{\ast}\in\mathbb{N}$ *such that,* $\forall\ n>N^{\ast}$, $m^{\infty}(z)$ *and* $m^{(n)}(z)$ *exist, and* $\|m^{(n)}_l(\cdot)-m^{\infty}_l(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Sigma)}\to 0$ *as* $n\to\infty$, $l=\pm1$.

A detailed exposition, including further motivations, for the construction of parametrices of the above-mentioned type can be found in [3, 57–59, 61, 90]; rather than regurgitating, verbatim, much of the analysis that can be found in the latter references, the point of view taken here is that one follows the scheme presented therein to obtain the results stated below, that is, the parametrix for the RHP $(\overset{e}{\mathcal{M}}{}^{\sharp}(z),\overset{e}{\upsilon}{}^{\sharp}(z),\Sigma^{\sharp}_{e})$ formulated in Lemma 4.2. In the case of the right-most end-points of the support of the 'even' equilibrium measure, $\{a^{e}_{j}\}_{j=1}^{N+1}$, a terse sketch of a proof is presented for the reader's convenience, and the remaining (left-most) end-points, namely, $b^{e}_{0},b^{e}_{1},\dots,b^{e}_{N}$, are analysed analogously.

The parametrix for the RHP $(\overset{e}{\mathcal{M}}{}^{\sharp}(z),\overset{e}{\upsilon}{}^{\sharp}(z),\Sigma^{\sharp}_{e})$ is now presented. By a parametrix of the RHP $(\overset{e}{\mathcal{M}}{}^{\sharp}(z),\overset{e}{\upsilon}{}^{\sharp}(z),\Sigma^{\sharp}_{e})$, in the neighbourhoods of the end-points of the support of the 'even' equilibrium measure, $\{b^{e}_{j-1},a^{e}_{j}\}_{j=1}^{N+1}$, is meant the solution of the RHPs formulated in the following two Lemmas (Lemmas 4.6 and 4.7). Define the 'small', mutually disjoint (open) discs about the end-points of the support of the 'even' equilibrium measure as follows: $\mathbb{U}^{e}_{\delta_{b_{j-1}}}:=\{z\in\mathbb{C};\ |z-b^{e}_{j-1}|<\delta^{e}_{b_{j-1}}\in(0,1)\}$ and $\mathbb{U}^{e}_{\delta_{a_j}}:=\{z\in\mathbb{C};\ |z-a^{e}_{j}|<\delta^{e}_{a_j}\in(0,1)\}$, $j=1,\dots,N+1$, where $\delta^{e}_{b_{j-1}}$ and $\delta^{e}_{a_j}$ are sufficiently small, positive real numbers chosen (amongst other things: see Lemmas 4.6 and 4.7 below) so that $\mathbb{U}^{e}_{\delta_{b_{i-1}}}\cap\mathbb{U}^{e}_{\delta_{a_j}}=\varnothing$, $i,j=1,\dots,N+1$ (the corresponding regions $\Omega^{e,l}_{b_{j-1}}$ and $\Omega^{e,l}_{a_j}$, and arcs $\Sigma^{e,l}_{b_{j-1}}$ and $\Sigma^{e,l}_{a_j}$, $j=1,\dots,N+1$, $l=1,2,3,4$, respectively, are defined more precisely below; see, also, Figures 5 and 6).

Remark 4.4. In order to simplify the results of Lemmas 4.6 and 4.7 (see below), it is convenient to introduce the following notation: (i)

$$\Psi^{e}_{1}(z):=\begin{pmatrix}\mathrm{Ai}(z) & \mathrm{Ai}(\omega^2 z)\\ \mathrm{Ai}'(z) & \omega^2\,\mathrm{Ai}'(\omega^2 z)\end{pmatrix}\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}\sigma_3},\qquad \Psi^{e}_{2}(z):=\begin{pmatrix}\mathrm{Ai}(z) & \mathrm{Ai}(\omega^2 z)\\ \mathrm{Ai}'(z) & \omega^2\,\mathrm{Ai}'(\omega^2 z)\end{pmatrix}\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}\sigma_3}(\mathrm{I}-\sigma_-),$$

$$\Psi^{e}_{3}(z):=\begin{pmatrix}\mathrm{Ai}(z) & -\omega^2\,\mathrm{Ai}(\omega z)\\ \mathrm{Ai}'(z) & -\mathrm{Ai}'(\omega z)\end{pmatrix}\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}\sigma_3}(\mathrm{I}+\sigma_-),\qquad \Psi^{e}_{4}(z):=\begin{pmatrix}\mathrm{Ai}(z) & -\omega^2\,\mathrm{Ai}(\omega z)\\ \mathrm{Ai}'(z) & -\mathrm{Ai}'(\omega z)\end{pmatrix}\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}\sigma_3},$$

where $\mathrm{Ai}(\cdot)$ is the Airy function (cf. Subsection 2.3), and $\omega=\exp(2\pi\mathrm{i}/3)$; and (ii)

$$\mho^{e}_{j}:=\begin{cases}\Omega^{e}_{j}, & j=1,\dots,N,\\ 0, & j=0,N+1,\end{cases}$$

where $\Omega^{e}_{j}=4\pi\int_{b^{e}_{j}}^{a^{e}_{N+1}}\psi^{e}_{V}(s)\,\mathrm{d}s$. ■

Lemma 4.6. *Let* $\overset{e}{\mathcal{M}}{}^{\sharp}\colon\mathbb{C}\setminus\Sigma^{\sharp}_{e}\to\mathrm{SL}_2(\mathbb{C})$ *solve the* RHP $(\overset{e}{\mathcal{M}}{}^{\sharp}(z),\overset{e}{\upsilon}{}^{\sharp}(z),\Sigma^{\sharp}_{e})$ *formulated in Lemma* 4.2*, and set*

$$\mathbb{U}^{e}_{\delta_{b_{j-1}}}:=\left\{z\in\mathbb{C};\ |z-b^{e}_{j-1}|<\delta^{e}_{b_{j-1}}\in(0,1)\right\},\qquad j=1,\dots,N+1.$$

Let

$$\Phi^{e}_{b_{j-1}}(z):=\left(\frac{3n}{4}\xi^{e}_{b_{j-1}}(z)\right)^{2/3},\qquad j=1,\dots,N+1,$$

with

$$\xi^{e}_{b_{j-1}}(z)=-2\int_{z}^{b^{e}_{j-1}}(R_e(s))^{1/2}h^{e}_{V}(s)\,\mathrm{d}s,$$

where, for $z\in\mathbb{U}^{e}_{\delta_{b_{j-1}}}\setminus(-\infty,b^{e}_{j-1})$, $\xi^{e}_{b_{j-1}}(z)=\mathfrak{b}(z-b^{e}_{j-1})^{3/2}G^{e}_{b_{j-1}}(z)$, $j=1,\dots,N+1$, *with* $\mathfrak{b}:=\pm1$ *for* $z\in\mathbb{C}_{\pm}$, *and* $G^{e}_{b_{j-1}}(z)$ *analytic, in particular,*

$$G^{e}_{b_{j-1}}(z)\underset{z\to b^{e}_{j-1}}{=}\frac{4}{3}f(b^{e}_{j-1})+\frac{4}{5}f'(b^{e}_{j-1})(z-b^{e}_{j-1})+\frac{2}{7}f''(b^{e}_{j-1})(z-b^{e}_{j-1})^2+\mathcal{O}\Big((z-b^{e}_{j-1})^3\Big),$$

where

$$\begin{aligned}
f(b_0^e) &= \mathrm{i}(-1)^N h_V^e(b_0^e)\eta_{b_0^e},\\
f'(b_0^e) &= \mathrm{i}(-1)^N\left(\frac{1}{2}h_V^e(b_0^e)\eta_{b_0^e}\left(\sum_{l=1}^N\left(\frac{1}{b_0^e-b_l^e}+\frac{1}{b_0^e-a_l^e}\right)+\frac{1}{b_0^e-a_{N+1}^e}\right)+(h_V^e(b_0^e))'\eta_{b_0^e}\right),\\
f''(b_0^e) &= \mathrm{i}(-1)^N\left(\frac{h_V^e(b_0^e)(h_V^e(b_0^e))''-((h_V^e(b_0^e))')^2}{h_V^e(b_0^e)}\eta_{b_0^e}-\frac{1}{2}h_V^e(b_0^e)\eta_{b_0^e}\right.\\
&\quad\times\left(\sum_{l=1}^N\left(\frac{1}{(b_0^e-b_l^e)^2}+\frac{1}{(b_0^e-a_l^e)^2}\right)+\frac{1}{(b_0^e-a_{N+1}^e)^2}\right)\\
&\quad+\left(\frac{1}{2}\left(\sum_{k=1}^N\left(\frac{1}{b_0^e-b_k^e}+\frac{1}{b_0^e-a_k^e}\right)+\frac{1}{b_0^e-a_{N+1}^e}\right)+\frac{(h_V^e(b_0^e))'}{h_V^e(b_0^e)}\right)\\
&\quad\times\left.\left(\frac{1}{2}h_V^e(b_0^e)\eta_{b_0^e}\left(\sum_{l=1}^N\left(\frac{1}{b_0^e-b_l^e}+\frac{1}{b_0^e-a_l^e}\right)+\frac{1}{b_0^e-a_{N+1}^e}\right)+(h_V^e(b_0^e))'\eta_{b_0^e}\right)\right),
\end{aligned}$$

with

$$\eta_{b_0^e}:=\left((a_{N+1}^e-b_0^e)\prod_{k=1}^N(b_k^e-b_0^e)(a_k^e-b_0^e)\right)^{1/2}\qquad(>0),$$

and, for $j=1,\dots,N$,

$$\begin{aligned}
f(b_j^e) &= \mathrm{i}(-1)^{N-j}h_V^e(b_j^e)\eta_{b_j^e},\\
f'(b_j^e) &= \mathrm{i}(-1)^{N-j}\left(\frac{1}{2}h_V^e(b_j^e)\eta_{b_j^e}\left(\sum_{\substack{k=1\\k\neq j}}^N\left(\frac{1}{b_j^e-b_k^e}+\frac{1}{b_j^e-a_k^e}\right)+\frac{1}{b_j^e-a_j^e}+\frac{1}{b_j^e-a_{N+1}^e}+\frac{1}{b_j^e-b_0^e}\right)\right.\\
&\quad+\left.(h_V^e(b_j^e))'\eta_{b_j^e}\right),\\
f''(b_j^e) &= \mathrm{i}(-1)^{N-j}\left(\frac{h_V^e(b_j^e)(h_V^e(b_j^e))''-((h_V^e(b_j^e))')^2}{h_V^e(b_j^e)}\eta_{b_j^e}-\frac{1}{2}h_V^e(b_j^e)\eta_{b_j^e}\left(\sum_{\substack{k=1\\k\neq j}}^N\left(\frac{1}{(b_j^e-b_k^e)^2}+\frac{1}{(b_j^e-a_k^e)^2}\right)\right.\right.\\
&\quad+\left.\frac{1}{(b_j^e-a_j^e)^2}+\frac{1}{(b_j^e-a_{N+1}^e)^2}+\frac{1}{(b_j^e-b_0^e)^2}\right)+\left(\frac{(h_V^e(b_j^e))'}{h_V^e(b_j^e)}+\frac{1}{2}\left(\sum_{\substack{k=1\\k\neq j}}^N\left(\frac{1}{b_j^e-b_k^e}+\frac{1}{b_j^e-a_k^e}\right)\right.\right.\\
&\quad+\left.\left.\frac{1}{b_j^e-a_j^e}+\frac{1}{b_j^e-a_{N+1}^e}+\frac{1}{b_j^e-b_0^e}\right)\right)\left(\frac{1}{2}h_V^e(b_j^e)\eta_{b_j^e}\left(\sum_{\substack{k=1\\k\neq j}}^N\left(\frac{1}{b_j^e-b_k^e}+\frac{1}{b_j^e-a_k^e}\right)\right.\right.\\
&\quad+\left.\left.\left.\frac{1}{b_j^e-a_j^e}+\frac{1}{b_j^e-a_{N+1}^e}+\frac{1}{b_j^e-b_0^e}\right)+(h_V^e(b_j^e))'\eta_{b_j^e}\right)\right),
\end{aligned}$$

with

$$\eta_{b_j^e}:=\left((b_j^e-a_j^e)(a_{N+1}^e-b_j^e)(b_j^e-b_0^e)\prod_{k=1}^{j-1}(b_j^e-b_k^e)(b_j^e-a_k^e)\prod_{l=j+1}^N(b_l^e-b_j^e)(a_l^e-b_j^e)\right)^{1/2}\qquad(>0),$$

and $((0,1)\ni)\ \delta_{b_{j-1}}^e$, $j=1,\dots,N+1$, *are chosen sufficiently small so that* $\Phi_{b_{j-1}}^e(z)$, *which are bi-holomorphic, conformal, and non-orientation preserving, map* $\mathbb{U}_{\delta_{b_{j-1}}}^e$ *(and, thus, the oriented contours* $\Sigma_{b_{j-1}}^e:=\cup_{l=1}^4\Sigma_{b_{j-1}}^{e,l}$, $j=1,\dots,N+1$: *Figure* 6) *injectively onto open* (n*-dependent) neighbourhoods* $\widehat{\mathbb{U}}_{\delta_{b_{j-1}}}^e$, $j=1,\dots,N+1$, *of* 0 *such that* $\Phi_{b_{j-1}}^e(b_{j-1}^e)=0$, $\Phi_{b_{j-1}}^e:\mathbb{U}_{\delta_{b_{j-1}}}^e\to\widehat{\mathbb{U}}_{\delta_{b_{j-1}}}^e:=\Phi_{b_{j-1}}^e(\mathbb{U}_{\delta_{b_{j-1}}}^e)$, $\Phi_{b_{j-1}}^e(\mathbb{U}_{\delta_{b_{j-1}}}^e\cap\Sigma_{b_{j-1}}^{e,l})=\Phi_{b_{j-1}}^e(\mathbb{U}_{\delta_{b_{j-1}}}^e)\cap\gamma_{b_{j-1}}^{e,l}$, *and*

$\Phi^{e}_{b_{j-1}}(\mathbb{U}^{e}_{\delta_{b_{j-1}}} \cap \Omega^{e,l}_{b_{j-1}}) = \Phi^{e}_{b_{j-1}}(\mathbb{U}^{e}_{\delta_{b_{j-1}}}) \cap \widehat{\Omega}^{e,l}_{b_{j-1}}$, *$l=1,2,3,4$, with* $\widehat{\Omega}^{e,1}_{b_{j-1}} = \{\zeta \in \mathbb{C};\ \arg(\zeta) \in (0,2\pi/3)\}$, $\widehat{\Omega}^{e,2}_{b_{j-1}} = \{\zeta \in \mathbb{C};\ \arg(\zeta) \in (2\pi/3,\pi)\}$, $\widehat{\Omega}^{e,3}_{b_{j-1}} = \{\zeta \in \mathbb{C};\ \arg(\zeta) \in (-\pi,-2\pi/3)\}$, *and* $\widehat{\Omega}^{e,4}_{b_{j-1}} = \{\zeta \in \mathbb{C};\ \arg(\zeta) \in (-2\pi/3,0)\}$.

The parametrix for the RHP $(\overset{e}{\mathcal{M}}{}^{\sharp}(z), \overset{e}{v}{}^{\sharp}(z), \Sigma^{\sharp}_{e})$, *for* $z \in \mathbb{U}^{e}_{\delta_{b_{j-1}}}$, *$j=1,\dots,N+1$, is the solution of the following* RHPs *for* $\mathcal{X}^{e} \colon \mathbb{U}^{e}_{\delta_{b_{j-1}}} \setminus \Sigma^{e}_{b_{j-1}} \to \mathrm{SL}_{2}(\mathbb{C})$, *$j=1,\dots,N+1$, where* $\Sigma^{e}_{b_{j-1}} := (\Phi^{e}_{b_{j-1}})^{-1}(\gamma^{e}_{b_{j-1}})$, *with* $(\Phi^{e}_{b_{j-1}})^{-1}$ *denoting the inverse mapping, and* $\gamma^{e}_{b_{j-1}} := \cup_{l=1}^{4} \gamma^{e,l}_{b_{j-1}}$: (i) *$\mathcal{X}^{e}(z)$ is holomorphic for* $z \in \mathbb{U}^{e}_{\delta_{b_{j-1}}} \setminus \Sigma^{e}_{b_{j-1}}$, *$j=1,\dots,N+1$;* (ii) $\mathcal{X}^{e}_{\pm}(z) := \lim_{\substack{z' \to z \\ z' \in \pm \text{ side of } \Sigma^{e}_{b_{j-1}}}} \mathcal{X}^{e}(z')$, *$j=1,\dots,N+1$, satisfy the boundary condition*

$$\mathcal{X}^{e}_{+}(z) = \mathcal{X}^{e}_{-}(z) \overset{e}{v}{}^{\sharp}(z), \quad z \in \mathbb{U}^{e}_{\delta_{b_{j-1}}} \cap \Sigma^{e}_{b_{j-1}}, \quad j=1,\dots,N+1,$$

where $\overset{e}{v}{}^{\sharp}(z)$ *is given in Lemma* 4.2; *and* (iii) *uniformly for* $z \in \partial \mathbb{U}^{e}_{\delta_{b_{j-1}}} := \left\{ z \in \mathbb{C};\ |z - b^{e}_{j-1}| = \delta^{e}_{b_{j-1}} \right\}$, *$j=1,\dots,N+1$,*

$$\overset{e}{m}{}^{\infty}(z)(\mathcal{X}^{e}(z))^{-1} \underset{\substack{n \to \infty \\ z \in \partial \mathbb{U}^{e}_{\delta_{b_{j-1}}}}}{=} \mathrm{I} + \mathcal{O}(n^{-1}), \quad j=1,\dots,N+1.$$

The solutions of the RHPs $(\mathcal{X}^{e}(z), \overset{e}{v}{}^{\sharp}(z), \mathbb{U}^{e}_{\delta_{b_{j-1}}} \cap \Sigma^{e}_{b_{j-1}})$, *$j=1,\dots,N+1$, are:*

(1) *for* $z \in \Omega^{e,1}_{b_{j-1}} := \mathbb{U}^{e}_{\delta_{b_{j-1}}} \cap (\Phi^{e}_{b_{j-1}})^{-1}(\widehat{\Omega}^{e,1}_{b_{j-1}})$, *$j=1,\dots,N+1$,*

$$\mathcal{X}^{e}(z) = \sqrt{\pi}\, \mathrm{e}^{-\frac{\mathrm{i}\pi}{3}} \overset{e}{m}{}^{\infty}(z) \sigma_{3} \mathrm{e}^{\frac{\mathrm{i}}{2} n \mho^{e}_{j-1} \operatorname{ad}(\sigma_{3})} \begin{pmatrix} \mathrm{i} & -\mathrm{i} \\ 1 & 1 \end{pmatrix} (\Phi^{e}_{b_{j-1}}(z))^{\frac{1}{4}\sigma_{3}} \Psi^{e}_{1}(\Phi^{e}_{b_{j-1}}(z)) \mathrm{e}^{\frac{1}{2} n \xi^{e}_{b_{j-1}}(z) \sigma_{3}} \sigma_{3},$$

where $\overset{e}{m}{}^{\infty}(z)$ *is given in Lemma* 4.5, *and* $\Psi^{e}_{1}(z)$ *and* $\mho^{e}_{k}$ *are defined in Remark* 4.4;

(2) *for* $z \in \Omega^{e,2}_{b_{j-1}} := \mathbb{U}^{e}_{\delta_{b_{j-1}}} \cap (\Phi^{e}_{b_{j-1}})^{-1}(\widehat{\Omega}^{e,2}_{b_{j-1}})$, *$j=1,\dots,N+1$,*

$$\mathcal{X}^{e}(z) = \sqrt{\pi}\, \mathrm{e}^{-\frac{\mathrm{i}\pi}{3}} \overset{e}{m}{}^{\infty}(z) \sigma_{3} \mathrm{e}^{\frac{\mathrm{i}}{2} n \mho^{e}_{j-1} \operatorname{ad}(\sigma_{3})} \begin{pmatrix} \mathrm{i} & -\mathrm{i} \\ 1 & 1 \end{pmatrix} (\Phi^{e}_{b_{j-1}}(z))^{\frac{1}{4}\sigma_{3}} \Psi^{e}_{2}(\Phi^{e}_{b_{j-1}}(z)) \mathrm{e}^{\frac{1}{2} n \xi^{e}_{b_{j-1}}(z) \sigma_{3}} \sigma_{3},$$

where $\Psi^{e}_{2}(z)$ *is defined in Remark* 4.4;

(3) *for* $z \in \Omega^{e,3}_{b_{j-1}} := \mathbb{U}^{e}_{\delta_{b_{j-1}}} \cap (\Phi^{e}_{b_{j-1}})^{-1}(\widehat{\Omega}^{e,3}_{b_{j-1}})$, *$j=1,\dots,N+1$,*

$$\mathcal{X}^{e}(z) = \sqrt{\pi}\, \mathrm{e}^{-\frac{\mathrm{i}\pi}{3}} \overset{e}{m}{}^{\infty}(z) \sigma_{3} \mathrm{e}^{-\frac{\mathrm{i}}{2} n \mho^{e}_{j-1} \operatorname{ad}(\sigma_{3})} \begin{pmatrix} \mathrm{i} & -\mathrm{i} \\ 1 & 1 \end{pmatrix} (\Phi^{e}_{b_{j-1}}(z))^{\frac{1}{4}\sigma_{3}} \Psi^{e}_{3}(\Phi^{e}_{b_{j-1}}(z)) \mathrm{e}^{\frac{1}{2} n \xi^{e}_{b_{j-1}}(z) \sigma_{3}} \sigma_{3},$$

where $\Psi^{e}_{3}(z)$ *is defined in Remark* 4.4;

(4) *for* $z \in \Omega^{e,4}_{b_{j-1}} := \mathbb{U}^{e}_{\delta_{b_{j-1}}} \cap (\Phi^{e}_{b_{j-1}})^{-1}(\widehat{\Omega}^{e,4}_{b_{j-1}})$, *$j=1,\dots,N+1$,*

$$\mathcal{X}^{e}(z) = \sqrt{\pi}\, \mathrm{e}^{-\frac{\mathrm{i}\pi}{3}} \overset{e}{m}{}^{\infty}(z) \sigma_{3} \mathrm{e}^{-\frac{\mathrm{i}}{2} n \mho^{e}_{j-1} \operatorname{ad}(\sigma_{3})} \begin{pmatrix} \mathrm{i} & -\mathrm{i} \\ 1 & 1 \end{pmatrix} (\Phi^{e}_{b_{j-1}}(z))^{\frac{1}{4}\sigma_{3}} \Psi^{e}_{4}(\Phi^{e}_{b_{j-1}}(z)) \mathrm{e}^{\frac{1}{2} n \xi^{e}_{b_{j-1}}(z) \sigma_{3}} \sigma_{3},$$

where $\Psi^{e}_{4}(z)$ *is defined in Remark* 4.4.

Lemma 4.7. *Let* $\overset{e}{\mathcal{M}}{}^{\sharp} \colon \mathbb{C} \setminus \Sigma^{\sharp}_{e} \to \mathrm{SL}_{2}(\mathbb{C})$ *solve the* RHP $(\overset{e}{\mathcal{M}}{}^{\sharp}(z), \overset{e}{v}{}^{\sharp}(z), \Sigma^{\sharp}_{e})$ *formulated in Lemma* 4.2, *and set*

$$\mathbb{U}^{e}_{\delta_{a_j}} := \left\{ z \in \mathbb{C};\ |z - a^{e}_{j}| < \delta^{e}_{a_j} \in (0,1) \right\}, \quad j=1,\dots,N+1.$$

Let

$$\Phi^{e}_{a_j}(z) := \left(\frac{3n}{4} \xi^{e}_{a_j}(z) \right)^{2/3}, \quad j=1,\dots,N+1,$$

with

$$\xi^{e}_{a_j}(z) = 2 \int_{a^{e}_{j}}^{z} (R_{e}(s))^{1/2} h^{e}_{V}(s)\, \mathrm{d}s,$$

where, for $z \in \mathbb{U}^e_{\delta_{a_j}} \setminus (-\infty, a^e_j)$, $\xi^e_{a_j}(z) = (z-a^e_j)^{3/2} G^e_{a_j}(z)$, $j=1,\dots,N+1$, *with* $G^e_{a_j}(z)$ *analytic, in particular,*

$$G^e_{a_j}(z) \underset{z \to a^e_j}{=} \frac{4}{3} f(a^e_j) + \frac{4}{5} f'(a^e_j)(z-a^e_j) + \frac{2}{7} f''(a^e_j)(z-a^e_j)^2 + \mathcal{O}\Big((z-a^e_j)^3\Big),$$

where

$$\begin{aligned}
f(a^e_{N+1}) &= h^e_V(a^e_{N+1}) \eta_{a^e_{N+1}}, \\
f'(a^e_{N+1}) &= \frac{1}{2} h^e_V(a^e_{N+1}) \eta_{a^e_{N+1}} \left(\sum_{l=1}^{N} \left(\frac{1}{a^e_{N+1}-b^e_l} + \frac{1}{a^e_{N+1}-a^e_l} \right) + \frac{1}{a^e_{N+1}-b^e_0} \right) \\
&\quad + (h^e_V(a^e_{N+1}))' \eta_{a^e_{N+1}}, \\
f''(a^e_{N+1}) &= \frac{h^e_V(a^e_{N+1})(h^e_V(a^e_{N+1}))'' - ((h^e_V(a^e_{N+1}))')^2}{h^e_V(a^e_{N+1})} \eta_{a^e_{N+1}} - \frac{1}{2} h^e_V(a^e_{N+1}) \eta_{a^e_{N+1}} \\
&\quad \times \left(\sum_{l=1}^{N} \left(\frac{1}{(a^e_{N+1}-b^e_l)^2} + \frac{1}{(a^e_{N+1}-a^e_l)^2} \right) + \frac{1}{(a^e_{N+1}-b^e_0)^2} \right) \\
&\quad + \left(\frac{1}{2} \left(\sum_{k=1}^{N} \left(\frac{1}{a^e_{N+1}-b^e_k} + \frac{1}{a^e_{N+1}-a^e_k} \right) + \frac{1}{a^e_{N+1}-b^e_0} \right) + \frac{(h^e_V(a^e_{N+1}))'}{h^e_V(a^e_{N+1})} \right) \\
&\quad \times \left(\frac{1}{2} h^e_V(a^e_{N+1}) \eta_{a^e_{N+1}} \left(\sum_{l=1}^{N} \left(\frac{1}{a^e_{N+1}-a^e_l} + \frac{1}{a^e_{N+1}-b^e_l} \right) + \frac{1}{a^e_{N+1}-b^e_0} \right) \right. \\
&\quad \left. + \ (h^e_V(a^e_{N+1}))' \eta_{a^e_{N+1}} \right),
\end{aligned}$$

with

$$\eta_{a^e_{N+1}} := \left((a^e_{N+1}-b^e_0) \prod_{k=1}^{N} (a^e_{N+1}-b^e_k)(a^e_{N+1}-a^e_k) \right)^{1/2} \qquad (>0),$$

and, for $j=1,\dots,N$,

$$\begin{aligned}
f(a^e_j) &= (-1)^{N-j+1} h^e_V(a^e_j) \eta_{a^e_j}, \\
f'(a^e_j) &= (-1)^{N-j+1} \left(\frac{1}{2} h^e_V(a^e_j) \eta_{a^e_j} \left(\sum_{\substack{k=1\\k\neq j}}^{N} \left(\frac{1}{a^e_j-b^e_k} + \frac{1}{a^e_j-a^e_k} \right) + \frac{1}{a^e_j-b^e_j} + \frac{1}{a^e_j-a^e_{N+1}} + \frac{1}{a^e_j-b^e_0} \right) \right. \\
&\quad \left. + \ (h^e_V(a^e_j))' \eta_{a^e_j} \right), \\
f''(a^e_j) &= (-1)^{N-j+1} \left(\frac{h^e_V(a^e_j)(h^e_V(a^e_j))'' - ((h^e_V(a^e_j))')^2}{h^e_V(a^e_j)} \eta_{a^e_j} - \frac{1}{2} h^e_V(a^e_j) \eta_{a^e_j} \left(\sum_{\substack{k=1\\k\neq j}}^{N} \left(\frac{1}{(a^e_j-b^e_k)^2} + \frac{1}{(a^e_j-a^e_k)^2} \right) \right. \right. \\
&\quad \left. + \ \frac{1}{(a^e_j-b^e_j)^2} + \frac{1}{(a^e_j-a^e_{N+1})^2} + \frac{1}{(a^e_j-b^e_0)^2} \right) + \left(\frac{(h^e_V(a^e_j))'}{h^e_V(a^e_j)} + \frac{1}{2} \left(\sum_{\substack{k=1\\k\neq j}}^{N} \left(\frac{1}{a^e_j-b^e_k} + \frac{1}{a^e_j-a^e_k} \right) \right. \right. \\
&\quad \left. \left. + \ \frac{1}{a^e_j-b^e_j} + \frac{1}{a^e_j-a^e_{N+1}} + \frac{1}{a^e_j-b^e_0} \right) \right) \left(\frac{1}{2} h^e_V(a^e_j) \eta_{a^e_j} \left(\sum_{\substack{k=1\\k\neq j}}^{N} \left(\frac{1}{a^e_j-b^e_k} + \frac{1}{a^e_j-a^e_k} \right) \right. \right. \\
&\quad \left. \left. \left. + \ \frac{1}{a^e_j-b^e_j} + \frac{1}{a^e_j-a^e_{N+1}} + \frac{1}{a^e_j-b^e_0} \right) + (h^e_V(a^e_j))' \eta_{a^e_j} \right) \right),
\end{aligned}$$

with

$$\eta_{a^e_j} := \left((b^e_j-a^e_j)(a^e_{N+1}-a^e_j)(a^e_j-b^e_0) \prod_{k=1}^{j-1} (a^e_j-b^e_k)(a^e_j-a^e_k) \prod_{l=j+1}^{N} (b^e_l-a^e_j)(a^e_l-a^e_j) \right)^{1/2} \qquad (>0),$$

and $((0,1)\ni)\,\delta^{e}_{a_j}$, $j=1,\dots,N+1$, *are chosen sufficiently small so that* $\Phi^{e}_{a_j}(z)$, *which are bi-holomorphic, conformal, and orientation preserving, map* $\mathbb{U}^{e}_{\delta_{a_j}}$ *(and, thus, the oriented contours* $\Sigma^{e}_{a_j}:=\cup_{l=1}^{4}\Sigma^{e,l}_{a_j}$, $j=1,\dots,N+1$: *Figure* 5) *injectively onto open (n-dependent) neighbourhoods* $\widehat{\mathbb{U}}^{e}_{\delta_{a_j}}$, $j=1,\dots,N+1$, *of* 0 *such that* $\Phi^{e}_{a_j}(a^{e}_{j})=0$, $\Phi^{e}_{a_j}:\mathbb{U}^{e}_{\delta_{a_j}}\to\widehat{\mathbb{U}}^{e}_{\delta_{a_j}}:=\Phi^{e}_{a_j}(\mathbb{U}^{e}_{\delta_{a_j}})$, $\Phi^{e}_{a_j}(\mathbb{U}^{e}_{\delta_{a_j}}\cap\Sigma^{e,l}_{a_j})=\Phi^{e}_{a_j}(\mathbb{U}^{e}_{\delta_{a_j}})\cap\gamma^{e,l}_{a_j}$, *and* $\Phi^{e}_{a_j}(\mathbb{U}^{e}_{\delta_{a_j}}\cap\Omega^{e,l}_{a_j})=\Phi^{e}_{a_j}(\mathbb{U}^{e}_{\delta_{a_j}})\cap\widehat{\Omega}^{e,l}_{a_j}$, $l=1,2,3,4$, *with* $\widehat{\Omega}^{e,1}_{a_j}=\{\zeta\in\mathbb{C};\ \arg(\zeta)\in(0,2\pi/3)\}$, $\widehat{\Omega}^{e,2}_{a_j}=\{\zeta\in\mathbb{C};\ \arg(\zeta)\in(2\pi/3,\pi)\}$, $\widehat{\Omega}^{e,3}_{a_j}=\{\zeta\in\mathbb{C};\ \arg(\zeta)\in(-\pi,-2\pi/3)\}$, *and* $\widehat{\Omega}^{e,4}_{a_j}=\{\zeta\in\mathbb{C};\ \arg(\zeta)\in(-2\pi/3,0)\}$.

The parametrix for the RHP $(\overset{e}{\mathcal{M}}{}^{\sharp}(z),\overset{e}{\upsilon}{}^{\sharp}(z),\Sigma^{\sharp}_{e})$, *for* $z\in\mathbb{U}^{e}_{\delta_{a_j}}$, $j=1,\dots,N+1$, *is the solution of the following* RHPs *for* $\mathcal{X}^{e}\colon\mathbb{U}^{e}_{\delta_{a_j}}\setminus\Sigma^{e}_{a_j}\to\mathrm{SL}_2(\mathbb{C})$, $j=1,\dots,N+1$, *where* $\Sigma^{e}_{a_j}:=(\Phi^{e}_{a_j})^{-1}(\gamma^{e}_{a_j})$, *with* $(\Phi^{e}_{a_j})^{-1}$ *denoting the inverse mapping, and* $\gamma^{e}_{a_j}:=\cup_{l=1}^{4}\gamma^{e,l}_{a_j}$: (i) $\mathcal{X}^{e}(z)$ *is holomorphic for* $z\in\mathbb{U}^{e}_{\delta_{a_j}}\setminus\Sigma^{e}_{a_j}$, $j=1,\dots,N+1$; (ii) $\mathcal{X}^{e}_{\pm}(z):=\lim_{\substack{z'\to z\\ z'\,\in\,\pm\,\text{side of}\,\Sigma^{e}_{a_j}}}\mathcal{X}^{e}(z')$, $j=1,\dots,N+1$, *satisfy the boundary condition*

$$\mathcal{X}^{e}_{+}(z)=\mathcal{X}^{e}_{-}(z)\overset{e}{\upsilon}{}^{\sharp}(z),\quad z\in\mathbb{U}^{e}_{\delta_{a_j}}\cap\Sigma^{e}_{a_j},\quad j=1,\dots,N+1,$$

where $\overset{e}{\upsilon}{}^{\sharp}(z)$ *is given in Lemma* 4.2; *and* (iii) *uniformly for* $z\in\partial\mathbb{U}^{e}_{\delta_{a_j}}:=\left\{z\in\mathbb{C};\ |z-a^{e}_{j}|=\delta^{e}_{a_j}\right\}$, $j=1,\dots,N+1$,

$$\overset{e}{m}{}^{\infty}(z)(\mathcal{X}^{e}(z))^{-1}\underset{\substack{n\to\infty\\ z\in\partial\mathbb{U}^{e}_{\delta_{a_j}}}}{=}\mathrm{I}+\mathcal{O}(n^{-1}),\quad j=1,\dots,N+1.$$

The solutions of the RHPs $(\mathcal{X}^{e}(z),\overset{e}{\upsilon}{}^{\sharp}(z),\mathbb{U}^{e}_{\delta_{a_j}}\cap\Sigma^{e}_{a_j})$, $j=1,\dots,N+1$, *are:*

(1) *for* $z\in\Omega^{e,1}_{a_j}:=\mathbb{U}^{e}_{\delta_{a_j}}\cap(\Phi^{e}_{a_j})^{-1}(\widehat{\Omega}^{e,1}_{a_j})$, $j=1,\dots,N+1$,

$$\mathcal{X}^{e}(z)=\sqrt{\pi}\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{3}}\overset{e}{m}{}^{\infty}(z)\mathrm{e}^{\frac{\mathrm{i}}{2}n\mho^{e}_{j}\,\mathrm{ad}(\sigma_3)}\begin{pmatrix}\mathrm{i}&-\mathrm{i}\\1&1\end{pmatrix}(\Phi^{e}_{a_j}(z))^{\frac{1}{4}\sigma_3}\Psi^{e}_{1}(\Phi^{e}_{a_j}(z))\mathrm{e}^{\frac{1}{2}n\xi^{e}_{a_j}(z)\sigma_3},$$

where $\overset{e}{m}{}^{\infty}(z)$ *is given in Lemma* 4.5, *and* $\Psi^{e}_{1}(z)$ *and* $\mho^{e}_{k}$ *are defined in Remark* 4.4;

(2) *for* $z\in\Omega^{e,2}_{a_j}:=\mathbb{U}^{e}_{\delta_{a_j}}\cap(\Phi^{e}_{a_j})^{-1}(\widehat{\Omega}^{e,2}_{a_j})$, $j=1,\dots,N+1$,

$$\mathcal{X}^{e}(z)=\sqrt{\pi}\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{3}}\overset{e}{m}{}^{\infty}(z)\mathrm{e}^{\frac{\mathrm{i}}{2}n\mho^{e}_{j}\,\mathrm{ad}(\sigma_3)}\begin{pmatrix}\mathrm{i}&-\mathrm{i}\\1&1\end{pmatrix}(\Phi^{e}_{a_j}(z))^{\frac{1}{4}\sigma_3}\Psi^{e}_{2}(\Phi^{e}_{a_j}(z))\mathrm{e}^{\frac{1}{2}n\xi^{e}_{a_j}(z)\sigma_3},$$

where $\Psi^{e}_{2}(z)$ *is defined in Remark* 4.4;

(3) *for* $z\in\Omega^{e,3}_{a_j}:=\mathbb{U}^{e}_{\delta_{a_j}}\cap(\Phi^{e}_{a_j})^{-1}(\widehat{\Omega}^{e,3}_{a_j})$, $j=1,\dots,N+1$,

$$\mathcal{X}^{e}(z)=\sqrt{\pi}\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{3}}\overset{e}{m}{}^{\infty}(z)\mathrm{e}^{-\frac{\mathrm{i}}{2}n\mho^{e}_{j}\,\mathrm{ad}(\sigma_3)}\begin{pmatrix}\mathrm{i}&-\mathrm{i}\\1&1\end{pmatrix}(\Phi^{e}_{a_j}(z))^{\frac{1}{4}\sigma_3}\Psi^{e}_{3}(\Phi^{e}_{a_j}(z))\mathrm{e}^{\frac{1}{2}n\xi^{e}_{a_j}(z)\sigma_3},$$

where $\Psi^{e}_{3}(z)$ *is defined in Remark* 4.4;

(4) *for* $z\in\Omega^{e,4}_{a_j}:=\mathbb{U}^{e}_{\delta_{a_j}}\cap(\Phi^{e}_{a_j})^{-1}(\widehat{\Omega}^{e,4}_{a_j})$, $j=1,\dots,N+1$,

$$\mathcal{X}^{e}(z)=\sqrt{\pi}\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{3}}\overset{e}{m}{}^{\infty}(z)\mathrm{e}^{-\frac{\mathrm{i}}{2}n\mho^{e}_{j}\,\mathrm{ad}(\sigma_3)}\begin{pmatrix}\mathrm{i}&-\mathrm{i}\\1&1\end{pmatrix}(\Phi^{e}_{a_j}(z))^{\frac{1}{4}\sigma_3}\Psi^{e}_{4}(\Phi^{e}_{a_j}(z))\mathrm{e}^{\frac{1}{2}n\xi^{e}_{a_j}(z)\sigma_3},$$

where $\Psi^{e}_{4}(z)$ *is defined in Remark* 4.4.

Remark 4.5. Perusing Lemmas 4.6 and 4.7, one notes that the normalisation condition at infinity, which is needed in order to guarantee the existence of solutions to the corresponding (parametrix) RHPs, is absent. The normalisation conditions at infinity are replaced by the (uniform) matching conditions $\overset{e}{m}{}^{\infty}(z)(\mathcal{X}^{e}(z))^{-1}\underset{\substack{n\to\infty\\ z\in\partial\mathbb{U}^{e}_{\delta_{*_j}}}}{=}\mathrm{I}+\mathcal{O}(n^{-1})$, where $*_j\in\{b_{j-1},a_j\}$, $j=1,\dots,N+1$, with $\partial\mathbb{U}^{e}_{\delta_{*_j}}$ defined in Lemmas 4.6 and 4.7. ■

Sketch of proof of Lemma 4.7. Let $(\overset{e}{\mathcal{M}}{}^{\sharp}(z),\overset{e}{\upsilon}{}^{\sharp}(z),\Sigma_e^{\sharp})$ be the RHP formulated in Lemma 4.2, and recall the definitions stated therein. For each $a_j^e\in\operatorname{supp}(\mu_V^e)$, $j=1,\dots,N+1$, define $\mathbb{U}_{\delta_{a_j}}^e$, $j=1,\dots,N+1$, as in the Lemma, that is, surround each right-most end-point a_j^e by open discs of radius $\delta_{a_j}^e\in(0,1)$ centred at a_j^e. Recalling the formula for $\overset{e}{\upsilon}{}^{\sharp}(z)$ given in Lemma 4.2, one shows, via the proof of Lemma 4.1, that:

(1) $4\pi\mathrm{i}\int_z^{a_{N+1}^e}\psi_V^e(s)\,\mathrm{d}s=4\pi\mathrm{i}(\int_z^{a_j^e}+\int_{a_j^e}^{b_j^e}+\int_{b_j^e}^{a_{N+1}^e})\psi_V^e(s)\,\mathrm{d}s$, whence, recalling the expression for the density of the 'even' equilibrium measure given in Lemma 3.5, that is, $\mathrm{d}\mu_V^e(x):=\psi_V^e(x)\,\mathrm{d}x=\frac{1}{2\pi\mathrm{i}}(R_e(x))_+^{1/2}h_V^e(x)\mathbf{1}_{J_e}(x)\,\mathrm{d}x$, one arrives at, upon considering the analytic continuation of $4\pi\mathrm{i}\cdot\int_z^{a_{N+1}^e}\psi_V^e(s)\,\mathrm{d}s$ to $\mathbb{C}\setminus\mathbb{R}$ (cf. proof of Lemma 4.1), in particular, to the oriented (open) skeletons $\mathbb{U}_{\delta_{a_j}}^e\cap(J_j^{e,\frown}\cup J_j^{e,\smile})$, $j=1,\dots,N+1$, $4\pi\mathrm{i}\int_z^{a_{N+1}^e}\psi_V^e(s)\,\mathrm{d}s=-\xi_{a_j}^e(z)+\mathrm{i}\mho_j^e$, $j=1,\dots,N+1$, where $\xi_{a_j}^e(z)=2\int_{a_j^e}^z(R_e(s))^{1/2}h_V^e(s)\,\mathrm{d}s$, and $\mho_j^e$ are defined in Remark 4.4;

(2) $g_+^e(z)+g_-^e(z)-\widetilde{V}(z)-\ell_e+2Q_e=-2\int_{a_j^e}^z(R_e(s))^{1/2}h_V^e(s)\,\mathrm{d}s<0$, $z\in(a_{N+1}^e,+\infty)\cup(\cup_{j=1}^N(a_j^e,b_j^e))$.

Via the latter formulae, which appear in the $(i\,j)$-elements, $i,j=1,2$, of the jump matrix $\overset{e}{\upsilon}{}^{\sharp}(z)$, denoting $\overset{e}{\mathcal{M}}{}^{\sharp}(z)$ by $\mathcal{X}^e(z)$ for $z\in\mathbb{U}_{\delta_{a_j}}^e$, $j=1,\dots,N+1$, and defining

$$\mathcal{P}_{a_j}^e(z):=\begin{cases}\mathcal{X}^e(z)\mathrm{e}^{-\frac{1}{2}n\xi_{a_j}^e(z)\sigma_3}\,\mathrm{e}^{\frac{\mathrm{i}}{2}n\mho_j^e\sigma_3}, & z\in\mathbb{C}_+\cap\mathbb{U}_{\delta_{a_j}}^e, \quad j=1,\dots,N+1,\\ \mathcal{X}^e(z)\mathrm{e}^{-\frac{1}{2}n\xi_{a_j}^e(z)\sigma_3}\,\mathrm{e}^{-\frac{\mathrm{i}}{2}n\mho_j^e\sigma_3}, & z\in\mathbb{C}_-\cap\mathbb{U}_{\delta_{a_j}}^e, \quad j=1,\dots,N+1,\end{cases}$$

one notes that $\mathcal{P}_{a_j}^e\colon\mathbb{U}_{\delta_{a_j}}^e\setminus J_{a_j}^e\to\mathrm{GL}_2(\mathbb{C})$, where $J_{a_j}^e:=J_j^{e,\frown}\cup J_j^{e,\smile}\cup(a_j^e-\delta_{a_j}^e,a_j^e+\delta_{a_j}^e)$, $j=1,\dots,N+1$, solve the RHPs $(\mathcal{P}_{a_j}^e(z),\upsilon_{\mathcal{P}_{a_j}}^e(z),J_{a_j}^e)$, with constant jump matrices $\upsilon_{\mathcal{P}_{a_j}}^e(z)$, $j=1,\dots,N+1$, defined by

$$\upsilon_{\mathcal{P}_{a_j}}^e(z):=\begin{cases}\mathrm{I}+\sigma_-, & z\in\mathbb{U}_{\delta_{a_j}}^e\cap(J_j^{e,\frown}\cup J_j^{e,\smile})=\Sigma_{a_j}^{e,1}\cup\Sigma_{a_j}^{e,3},\\ \mathrm{I}+\sigma_+, & z\in\mathbb{U}_{\delta_{a_j}}^e\cap(a_j^e,a_j^e+\delta_{a_j}^e)=\Sigma_{a_j}^{e,4},\\ \mathrm{i}\sigma_2, & z\in\mathbb{U}_{\delta_{a_j}}^e\cap(a_j^e-\delta_{a_j}^e,a_j^e)=\Sigma_{a_j}^{e,2},\end{cases}$$

subject, still, to the asymptotic matching conditions $\overset{e}{m}{}^{\infty}(z)(\mathcal{X}^e(z))^{-1}=_{n\to\infty}\mathrm{I}+\mathcal{O}(n^{-1})$, uniformly for $z\in\partial\mathbb{U}_{\delta_{a_j}}^e$, $j=1,\dots,N+1$.

Set, as in the Lemma, $\Phi_{a_j}^e(z):=(\frac{3}{4}n\xi_{a_j}^e(z))^{2/3}$, $j=1,\dots,N+1$, with $\xi_{a_j}^e(z)$ defined above: a careful analysis of the branch cuts shows that, for $z\in\mathbb{U}_{\delta_{a_j}}^e$, $j=1,\dots,N+1$, $\Phi_{a_j}^e(z)$ and $\xi_{a_j}^e(z)$ satisfy the properties stated in the Lemma; in particular, for $\Phi_{a_j}^e\colon\mathbb{U}_{\delta_{a_j}}^e\to\mathbb{C}$, $j=1,\dots,N+1$, $\Phi_{a_j}^e(z)=(z-a_j^e)^{3/2}G_{a_j}^e(z)$, with $G_{a_j}^e(z)$ holomorphic for $z\in\mathbb{U}_{\delta_{a_j}}^e$ and characterised in the Lemma, $\Phi_{a_j}^e(a_j^e)=0$, $(\Phi_{a_j}^e(z))'\neq0$, $z\in\mathbb{U}_{\delta_{a_j}}^e$, and where $(\Phi_{a_j}^e(a_j^e))'=(nf(a_j^e))^{2/3}>0$, with $f(a_j^e)$ given in the Lemma. One now chooses $\delta_{a_j}^e$ $(\in(0,1))$, $j=1,\dots,N+1$, and the oriented—open—skeletons ('near' a_j^e) J_{a_j}, $j=1,\dots,N+1$, in such a way that their images under the bi-holomorphic, conformal, and orientation-preserving mappings $\Phi_{a_j}^e(z)$ are the union of the straight-line segments $\gamma_{a_j}^{e,l}$, $l=1,2,3,4$, $j=1,\dots,N+1$. Set $\zeta:=\Phi_{a_j}^e(z)$, $j=1,\dots,N+1$, and consider $\mathcal{X}^e(\Phi_{a_j}^e(z)):=\Psi^e(\zeta)$. Recalling the properties of $\Phi_{a_j}^e(z)$, a straightforward calculation shows that $\Psi^e\colon\Phi_{a_j}^e(\mathbb{U}_{\delta_{a_j}}^e)\setminus\cup_{l=1}^4\gamma_{a_j}^{e,l}\to\mathrm{GL}_2(\mathbb{C})$, $j=1,\dots,N+1$, solves the RHPs $(\Psi^e(\zeta),\upsilon_{\Psi^e}^e(\zeta),\cup_{l=1}^4\gamma_{a_j}^{e,l})$, $j=1,\dots,N+1$, with constant jump matrices $\upsilon_{\Psi^e}^e(\zeta)$, $j=1,\dots,N+1$, defined by

$$\upsilon_{\Psi^e}^e(\zeta):=\begin{cases}\mathrm{I}+\sigma_-, & \zeta\in\gamma_{a_j}^{e,1}\cup\gamma_{a_j}^{e,3},\\ \mathrm{I}+\sigma_+, & \zeta\in\gamma_{a_j}^{e,4},\\ \mathrm{i}\sigma_2, & \zeta\in\gamma_{a_j}^{e,2}.\end{cases}$$

The solution of the latter (yet-to-be normalised) RHPs is well known; in fact, their solution is expressed

in terms of the Airy function, and is given by (see, for example, [3, 58, 59, 61, 90])

$$\Psi^e(\zeta)=\begin{cases}\Psi_1^e(\zeta), & \zeta\in\widehat{\Omega}_{a_j}^{e,1}, \quad j=1,\dots,N+1,\\ \Psi_2^e(\zeta), & \zeta\in\widehat{\Omega}_{a_j}^{e,2}, \quad j=1,\dots,N+1,\\ \Psi_3^e(\zeta), & \zeta\in\widehat{\Omega}_{a_j}^{e,3}, \quad j=1,\dots,N+1,\\ \Psi_4^e(\zeta), & \zeta\in\widehat{\Omega}_{a_j}^{e,4}, \quad j=1,\dots,N+1,\end{cases}$$

where $\Psi_k^e(z)$, $k=1,2,3,4$, are defined in Remark 4.4. Recalling that $\Phi_{a_j}^e(z)$, $j=1,\dots,N+1$, are biholomorphic and orientation-preserving conformal mappings, with $\Phi_{a_j}^e(a_j^e)=0$ and $\Phi_{a_j}^e$ $(:\mathbb{U}_{\delta_{a_j}}^e\to\Phi_{a_j}^e(\mathbb{U}_{\delta_{a_j}}^e)=:\widehat{\mathbb{U}}_{\delta_{a_j}}^e)$: $\mathbb{U}_{\delta_{a_j}}^e\cap J_{a_j}^e\to\Phi_{a_j}^e(\mathbb{U}_{\delta_{a_j}}^e\cap J_{a_j}^e)=\widehat{\mathbb{U}}_{\delta_{a_j}}^e\cap(\cup_{l=1}^4\gamma_{a_j}^{e,l})$, $j=1,\dots,N+1$, one notes that, for any analytic maps $E_{a_j}^e:\mathbb{U}_{\delta_{a_j}}^e\to\mathrm{GL}_2(\mathbb{C})$, $j=1,\dots,N+1$, $\mathbb{U}_{\delta_{a_j}}^e\setminus J_{a_j}^e\ni\zeta\mapsto E_{a_j}^e(\zeta)\Psi^e(\zeta)$ also solves the latter RHPs $(\Psi^e(\zeta),\upsilon_{\Psi^e}^e(\zeta),\cup_{l=1}^4\gamma_{a_j}^{e,l})$, $j=1,\dots,N+1$: one uses this 'degree of freedom' of 'multiplying on the left' by a non-degenerate, analytic, matrix-valued function in order to satisfy the remaining asymptotic (as $n\to\infty$) matching condition for the parametrix, namely, $\overset{e}{m}{}^{\infty}(z)(\mathcal{X}^e(z))^{-1}\underset{\substack{n\to\infty\\ z\in\partial\mathbb{U}_{\delta_{a_j}}^e}}{=}\mathrm{I}+\mathcal{O}(n^{-1})$, uniformly for $z\in\partial\mathbb{U}_{\delta_{a_j}}^e$, $j=1,\dots,N+1$.

Consider, say, and without loss of generality, the regions $\Omega_{a_j}^{e,1}:=(\Phi_{a_j}^e)^{-1}(\widehat{\Omega}_{a_j}^{e,1})$, $j=1,\dots,N+1$ (Figure 5). Re-tracing the above transformations, one shows that, for $z\in\Omega_{a_j}^{e,1}$ $(\subset\mathbb{C}_+)$, $j=1,\dots,N+1$, $\mathcal{X}^e(z)=E_{a_j}^e(z)\Psi^e((\frac{3}{4}n\xi_{a_j}^e(z))^{2/3})\exp(\frac{n}{2}(\xi_{a_j}^e(z)-\mathrm{i}\mho_j^e)\sigma_3)$, whence, using the expression above for $\Psi^e(\zeta)$, $\zeta\in\mathbb{C}_+\cap\widehat{\Omega}_{a_j}^{e,1}$, $j=1,\dots,N+1$, and the asymptotic expansions for $\mathrm{Ai}(\cdot)$ and $\mathrm{Ai}'(\cdot)$ (as $n\to\infty$) given in Equations (2.6), one arrives at

$$\mathcal{X}^e(z)\underset{\substack{n\to\infty\\ z\in\partial\Omega_{a_j}^{e,1}\cap\partial\mathbb{U}_{\delta_{a_j}}^e}}{=}\frac{1}{\sqrt{2\pi}}E_{a_j}^e(z)\left(\left(\frac{3}{4}n\xi_{a_j}^e(z)\right)^{2/3}\right)^{-\frac{1}{4}\sigma_3}\begin{pmatrix}\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}} & \mathrm{e}^{\frac{\mathrm{i}\pi}{3}}\\ -\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}} & -\mathrm{e}^{\frac{4\pi\mathrm{i}}{3}}\end{pmatrix}\mathrm{e}^{-\frac{\mathrm{i}}{2}n\mho_j^e\sigma_3}\big(\mathrm{I}+\mathcal{O}(n^{-1})\big):$$

demanding that, for $z\in\partial\Omega_{a_j}^{e,1}\cap\partial\mathbb{U}_{\delta_{a_j}}^e$, $j=1,\dots,N+1$, $\overset{e}{m}{}^{\infty}(z)(\mathcal{X}^e(z))^{-1}=_{n\to\infty}\mathrm{I}+\mathcal{O}(n^{-1})$, one gets that

$$E_{a_j}^e(z)=\frac{1}{\sqrt{2\mathrm{i}}}\overset{e}{m}{}^{\infty}(z)\mathrm{e}^{\frac{\mathrm{i}}{2}n\mho_j^e\sigma_3}\begin{pmatrix}\mathrm{i} & -\mathrm{i}\\ 1 & 1\end{pmatrix}\left(\left(\frac{3}{4}n\xi_{a_j}^e(z)\right)^{2/3}\right)^{\frac{1}{4}\sigma_3}, \quad j=1,\dots,N+1$$

(note that $\det(E_{a_j}^e(z))=1$). One mimicks the above paradigm for the remaining boundary skeletons $\partial\Omega_{a_j}^{e,l}\cap\partial\mathbb{U}_{\delta_{a_j}}^e$, $l=2,3,4$, $j=1,\dots,N+1$, and shows that the exact same formula for $E_{a_j}^e(z)$ given above is obtained; thus, for $E_{a_j}^e(z)$, $j=1,\dots,N+1$, as given above, one concludes that, uniformly for $z\in\partial\mathbb{U}_{\delta_{a_j}}^e$, $j=1,\dots,N+1$, $\overset{e}{m}{}^{\infty}(z)(\mathcal{X}^e(z))^{-1}\underset{\substack{n\to\infty\\ z\in\partial\mathbb{U}_{\delta_{a_j}}^e}}{=}\mathrm{I}+\mathcal{O}(n^{-1})$. There remains, however, the question of unimodularity, since

$$\det(\mathcal{X}^e(z))=\begin{vmatrix}\mathrm{Ai}(\Phi_{a_j}^e(z)) & \mathrm{Ai}(\omega^2\Phi_{a_j}^e(z))\\ \mathrm{Ai}'(\Phi_{a_j}^e(z)) & \omega^2\,\mathrm{Ai}'(\omega^2\Phi_{a_j}^e(z))\end{vmatrix}\quad\text{or}\quad\begin{vmatrix}\mathrm{Ai}(\Phi_{a_j}^e(z)) & -\omega^2\,\mathrm{Ai}(\omega\Phi_{a_j}^e(z))\\ \mathrm{Ai}'(\Phi_{a_j}^e(z)) & -\mathrm{Ai}'(\omega\Phi_{a_j}^e(z))\end{vmatrix}:$$

multiplying $\mathcal{X}^e(z)$ on the left by a constant, $\widetilde{c}$, say, using the Wronskian relations (see Chapter 10 of [93]) $\mathrm{W}(\mathrm{Ai}(\lambda),\mathrm{Ai}(\omega^2\lambda))=(2\pi)^{-1}\exp(\mathrm{i}\pi/6)$ and $\mathrm{W}(\mathrm{Ai}(\lambda),\mathrm{Ai}(\omega\lambda))=-(2\pi)^{-1}\exp(-\mathrm{i}\pi/6)$, and the linear dependence relation for Airy functions, $\mathrm{Ai}(\lambda)+\omega\,\mathrm{Ai}(\omega\lambda)+\omega^2\,\mathrm{Ai}(\omega^2\lambda)=0$, one shows that, upon imposing the condition $\det(\mathcal{X}^e(z))=1$, $\widetilde{c}=(2\pi)^{1/2}\exp(-\mathrm{i}\pi/12)$. □

The above analyses lead to the following lemma.

Lemma 4.8. *Let* $\overset{e}{\mathcal{M}}{}^{\sharp}:\mathbb{C}\setminus\Sigma_e^{\sharp}\to\mathrm{SL}_2(\mathbb{C})$ *solve the* RHP $(\overset{e}{\mathcal{M}}{}^{\sharp}(z),\overset{e}{\upsilon}{}^{\sharp}(z),\Sigma_e^{\sharp})$ *formulated in Lemma* 4.2. *Define*

$$\mathcal{S}_p^e(z):=\begin{cases}\overset{e}{m}{}^{\infty}(z), & z\in\mathbb{C}\setminus\cup_{j=1}^{N+1}(\mathbb{U}_{\delta_{b_{j-1}}}^e\cup\mathbb{U}_{\delta_{a_j}}^e),\\ \mathcal{X}^e(z), & z\in\cup_{j=1}^{N+1}(\mathbb{U}_{\delta_{b_{j-1}}}^e\cup\mathbb{U}_{\delta_{a_j}}^e),\end{cases}$$

where $\overset{e}{m}^{\infty} \colon \mathbb{C} \setminus J_e^{\infty} \to \mathrm{SL}_2(\mathbb{C})$ *is characterised completely in Lemma* 4.5, *and:* (1) *for* $z \in \mathbb{U}^e_{\delta_{b_{j-1}}}$, $j=1,\dots,N+1$, $\mathcal{X}^e \colon \mathbb{U}^e_{\delta_{b_{j-1}}} \setminus \Sigma^e_{b_{j-1}} \to \mathrm{SL}_2(\mathbb{C})$ *solve the* RHPs $(\mathcal{X}^e(z), \overset{e}{\upsilon}{}^{\sharp}(z), \Sigma^e_{b_{j-1}})$, $j=1,\dots,N+1$, *formulated in Lemma* 4.6; *and* (2) *for* $z \in \mathbb{U}^e_{\delta_{a_j}}$, $j=1,\dots,N+1$, $\mathcal{X}^e \colon \mathbb{U}^e_{\delta_{a_j}} \setminus \Sigma^e_{a_j} \to \mathrm{SL}_2(\mathbb{C})$ *solve the* RHPs $(\mathcal{X}^e(z), \overset{e}{\upsilon}{}^{\sharp}(z), \Sigma^e_{a_j})$, $j=1,\dots,N+1$, *formulated in Lemma* 4.7. *Set*

$$\mathcal{R}^e(z) := \overset{e}{\mathcal{M}}{}^{\sharp}(z)\big(\mathcal{S}^e_p(z)\big)^{-1},$$

and define the augmented contour $\Sigma^e_p := \Sigma^{\sharp}_e \cup (\cup_{j=1}^{N+1}(\partial\mathbb{U}^e_{\delta_{b_{j-1}}} \cup \mathbb{U}^e_{\delta_{a_j}}))$, *with the orientation given in Figure* 9. *Then* $\mathcal{R}^e \colon \mathbb{C} \setminus \Sigma^e_p \to \mathrm{SL}_2(\mathbb{C})$ *solves the following* RHP: (i) $\mathcal{R}^e(z)$ *is holomorphic for* $z \in \mathbb{C} \setminus \Sigma^e_p$; (ii) $\mathcal{R}^e_{\pm}(z) := \lim_{\substack{z' \to z \\ z' \in \pm \text{ side of } \Sigma^e_p}} \mathcal{R}^e(z')$ *satisfy the boundary condition*

$$\mathcal{R}^e_+(z) = \mathcal{R}^e_-(z)\upsilon^e_{\mathcal{R}}(z), \quad z \in \Sigma^e_p,$$

where

$$\upsilon^e_{\mathcal{R}}(z) := \begin{cases} \upsilon^{e,1}_{\mathcal{R}}(z), & z \in (-\infty, b^e_0 - \delta^e_{b_0}) \cup (a^e_{N+1} + \delta^e_{a_{N+1}}, +\infty) =: \Sigma^{e,1}_p, \\ \upsilon^{e,2}_{\mathcal{R}}(z), & z \in (a^e_j + \delta^e_{a_j}, b^e_j - \delta^e_{b_j}) =: \Sigma^{e,2}_{p,j} \subset \cup_{l=1}^{N} \Sigma^{e,2}_{p,l} =: \Sigma^{e,2}_p, \\ \upsilon^{e,3}_{\mathcal{R}}(z), & z \in \cup_{j=1}^{N+1}(J^{e,\frown}_j \setminus (\mathbb{C}_+ \cap (\mathbb{U}^e_{\delta_{b_{j-1}}} \cup \mathbb{U}^e_{\delta_{a_j}}))) =: \Sigma^{e,3}_p, \\ \upsilon^{e,4}_{\mathcal{R}}(z), & z \in \cup_{j=1}^{N+1}(J^{e,\smile}_j \setminus (\mathbb{C}_- \cap (\mathbb{U}^e_{\delta_{b_{j-1}}} \cup \mathbb{U}^e_{\delta_{a_j}}))) =: \Sigma^{e,4}_p, \\ \upsilon^{e,5}_{\mathcal{R}}(z), & z \in \cup_{j=1}^{N+1}(\partial\mathbb{U}^e_{\delta_{b_{j-1}}} \cup \mathbb{U}^e_{\delta_{a_j}}) =: \Sigma^{e,5}_p, \\ \mathrm{I}, & z \in \Sigma^e_p \setminus \cup_{l=1}^{5} \Sigma^{e,l}_p, \end{cases}$$

with

$$\begin{aligned}
\upsilon^{e,1}_{\mathcal{R}}(z) &= \mathrm{I} + \mathrm{e}^{n(g^e_+(z)+g^e_-(z)-\widetilde{V}(z)-\ell_e+2Q_e)}\,\overset{e}{m}^{\infty}(z)\sigma_+(\overset{e}{m}^{\infty}(z))^{-1}, \\
\upsilon^{e,2}_{\mathcal{R}}(z) &= \mathrm{I} + \mathrm{e}^{-\mathrm{i}n\Omega^e_j+n(g^e_+(z)+g^e_-(z)-\widetilde{V}(z)-\ell_e+2Q_e)}\,\overset{e}{m}^{\infty}_-(z)\sigma_+(\overset{e}{m}^{\infty}_-(z))^{-1}, \\
\upsilon^{e,3}_{\mathcal{R}}(z) &= \mathrm{I} + \mathrm{e}^{-4n\pi\mathrm{i}\int_z^{a^e_{N+1}} \psi^e_V(s)\,\mathrm{d}s}\,\overset{e}{m}^{\infty}(z)\sigma_-(\overset{e}{m}^{\infty}(z))^{-1}, \\
\upsilon^{e,4}_{\mathcal{R}}(z) &= \mathrm{I} + \mathrm{e}^{4n\pi\mathrm{i}\int_z^{a^e_{N+1}} \psi^e_V(s)\,\mathrm{d}s}\,\overset{e}{m}^{\infty}(z)\sigma_-(\overset{e}{m}^{\infty}(z))^{-1}, \\
\upsilon^{e,5}_{\mathcal{R}}(z) &= \mathcal{X}^e(z)(\overset{e}{m}^{\infty}(z))^{-1}:
\end{aligned}$$

(iii) $\mathcal{R}^e(z) \underset{\substack{z \to \infty \\ z \in \mathbb{C}\setminus\Sigma^e_p}}{=} \mathrm{I} + \mathcal{O}(z^{-1})$; *and* (iv) $\mathcal{R}^e(z) \underset{\substack{z \to 0 \\ z \in \mathbb{C}\setminus\Sigma^e_p}}{=} \mathcal{O}(1)$.

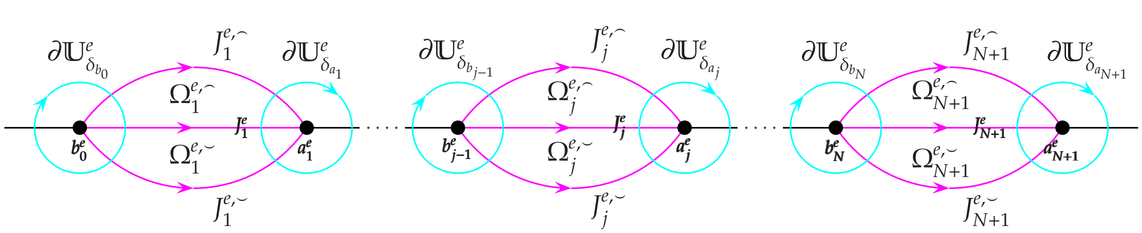

Figure 9: The augmented contour $\Sigma^e_p := \Sigma^{\sharp}_e \cup (\cup_{j=1}^{N+1}(\partial\mathbb{U}^e_{\delta_{b_{j-1}}} \cup \partial\mathbb{U}^e_{\delta_{a_j}}))$

Proof. Define the oriented, augmented skeleton Σ^e_p as in the Lemma: the RHP $(\mathcal{R}^e(z), \upsilon^e_{\mathcal{R}}(z), \Sigma^e_p)$ follows from the RHPs $(\overset{e}{\mathcal{M}}{}^{\sharp}(z), \overset{e}{\upsilon}{}^{\sharp}(z), \Sigma^{\sharp}_e)$ and $(\overset{e}{m}^{\infty}(z), \overset{e}{\upsilon}{}^{\infty}(z), J^{\infty}_e)$ formulated in Lemmas 4.2 and 4.3, respectively, upon using the definitions of $\mathcal{S}^e_p(z)$ and $\mathcal{R}^e(z)$ given in the Lemma. □

5 Asymptotic (as $n \to \infty$) Solution of the RHP for $\overset{e}{\mathrm{Y}}(z)$

In this section, via the Beals-Coifman (BC) construction [84], the (normalised at infinity) RHP ($\mathcal{R}^e(z)$, $\upsilon_{\mathcal{R}}^e(z)$, Σ_p^e) formulated in Lemma 4.8 is solved asymptotically (as $n \to \infty$); in particular, it is shown that, uniformly for $z \in \Sigma_p^e$,

$$\|\upsilon_{\mathcal{R}}^e(\cdot) - \mathrm{I}\|_{\cap_{p \in \{1,2,\infty\}} \mathcal{L}^p_{\mathrm{M}_2(\mathbb{C})}(\Sigma_p^e)} \underset{n \to \infty}{=} \mathrm{I} + \mathcal{O}\big(f(n) n^{-1}\big),$$

where $f(n) =_{n \to \infty} \mathcal{O}(1)$, and, subsequently, the original **RHP1**, that is, $(\overset{e}{\mathrm{Y}}(z), \mathrm{I} + \mathrm{e}^{-n\widetilde{V}(z)}\sigma_+, \mathbb{R})$, is solved asymptotically by re-tracing the finite sequence of RHP transformations $\mathcal{R}^e(z)$ (Lemmas 5.3 and 4.8) $\to \overset{e}{\mathcal{M}}{}^\sharp(z)$ (Lemma 4.2) $\to \overset{e}{\mathcal{M}}(z)$ (Lemma 3.4) $\to \overset{e}{\mathrm{Y}}(z)$. The (unique) solution for $\overset{e}{\mathrm{Y}}(z)$ then leads to the final asymptotic results for $\boldsymbol{\pi}_{2n}(z)$ (in the entire complex plane), $\xi_n^{(2n)}$ and $\phi_{2n}(z)$ (in the entire complex plane) stated, respectively, in Theorems 2.3.1 and 2.3.2.

Proposition 5.1. *Let* $\mathcal{R}^e \colon \mathbb{C} \setminus \Sigma_p^e \to \mathrm{SL}_2(\mathbb{C})$ *solve the* RHP ($\mathcal{R}^e(z)$, $\upsilon_{\mathcal{R}}^e(z)$, Σ_p^e) *formulated in Lemma* 4.8*. Then:*

(1) *for* $z \in (-\infty, b_0^e - \delta_{b_0}^e) \cup (a_{N+1}^e + \delta_{a_{N+1}}^e, +\infty) =: \Sigma_p^{e,1}$,

$$\upsilon_{\mathcal{R}}^e(z) \underset{n \to \infty}{=} \begin{cases} \mathrm{I} + \mathcal{O}(f_\infty(n) \mathrm{e}^{-nc_\infty |z|}), & z \in \Sigma_p^{e,1} \setminus \mathbb{U}_0^e, \\ \mathrm{I} + \mathcal{O}(f_0(n) \mathrm{e}^{-nc_0 |z|^{-1}}), & z \in \Sigma_p^e \cap \mathbb{U}_0^e, \end{cases}$$

where $c_0, c_\infty > 0$, $(f_\infty(n))_{ij} =_{n \to \infty} \mathcal{O}(1)$, $(f_0(n))_{ij} =_{n \to \infty} \mathcal{O}(1)$, $i, j = 1, 2$, *and* $\mathbb{U}_0^e := \{z \in \mathbb{C};\ |z| < \epsilon\}$, *with* ϵ *some arbitrarily fixed, sufficiently small positive real number;*

(2) *for* $z \in (a_j^e + \delta_{a_j}^e, b_j^e - \delta_{b_j}^e) =: \Sigma_{p,j}^{e,2} \subset \cup_{l=1}^N \Sigma_{p,l}^{e,2} =: \Sigma_p^{e,2}$, $j = 1, \dots, N$,

$$\upsilon_{\mathcal{R}}^e(z) \underset{n \to \infty}{=} \begin{cases} \mathrm{I} + \mathcal{O}(f_j(n) \mathrm{e}^{-nc_j (z - a_j^e)}), & z \in \Sigma_{p,j}^{e,2} \setminus \mathbb{U}_0^e, \\ \mathrm{I} + \mathcal{O}(\widetilde{f}_j(n) \mathrm{e}^{-n\widetilde{c}_j |z|^{-1}}), & z \in \Sigma_{p,j}^{e,2} \cap \mathbb{U}_0^e, \end{cases}$$

where $c_j, \widetilde{c}_j > 0$, $(f_j(n))_{kl} =_{n \to \infty} \mathcal{O}(1)$, *and* $(\widetilde{f}_j(n))_{kl} =_{n \to \infty} \mathcal{O}(1)$, $k, l = 1, 2$;

(3) *for* $z \in \cup_{j=1}^{N+1} (J_j^{e,\frown} \setminus (\mathbb{C}_+ \cap (\mathbb{U}^e_{\delta_{b_{j-1}}} \cup \mathbb{U}^e_{\delta_{a_j}}))) =: \Sigma_p^{e,3}$,

$$\upsilon_{\mathcal{R}}^e(z) \underset{n \to \infty}{=} \mathrm{I} + \mathcal{O}(\widehat{f}(n) \mathrm{e}^{-n\widehat{c}|z|}),$$

where $\widehat{c} > 0$ *and* $(\widehat{f}(n))_{ij} =_{n \to \infty} \mathcal{O}(1)$, $i, j = 1, 2$;

(4) *for* $z \in \cup_{j=1}^{N+1} (J_j^{e,\smile} \setminus (\mathbb{C}_- \cap (\mathbb{U}^e_{\delta_{b_{j-1}}} \cup \mathbb{U}^e_{\delta_{a_j}}))) =: \Sigma_p^{e,4}$,

$$\upsilon_{\mathcal{R}}^e(z) \underset{n \to \infty}{=} \mathrm{I} + \mathcal{O}(\breve{f}(n) \mathrm{e}^{-n\breve{c}|z|}),$$

where $\breve{c} > 0$ *and* $(\breve{f}(n))_{ij} =_{n \to \infty} \mathcal{O}(1)$, $i, j = 1, 2$; *and*

(5) *for* $z \in \cup_{j=1}^{N+1} (\partial \mathbb{U}^e_{\delta_{b_{j-1}}} \cup \partial \mathbb{U}^e_{\delta_{a_j}}) =: \Sigma_p^{e,5}$,

$$\begin{aligned} \upsilon_{\mathcal{R}}^e(z) \underset{\substack{n \to \infty \\ z \in \mathbb{C}_\pm \cap \partial \mathbb{U}^e_{\delta_{b_{j-1}}}}}{=} & \mathrm{I} + \frac{1}{n \xi^e_{b_{j-1}}(z)} \overset{e}{\mathfrak{M}}{}^\infty(z) \begin{pmatrix} \mp(s_1 + t_1) & \mp\mathrm{i}(s_1 - t_1)\mathrm{e}^{\mathrm{i}n\mho^e_{j-1}} \\ \mp\mathrm{i}(s_1 - t_1)\mathrm{e}^{-\mathrm{i}n\mho^e_{j-1}} & \pm(s_1 + t_1) \end{pmatrix} (\overset{e}{\mathfrak{M}}{}^\infty(z))^{-1} \\ & + \mathcal{O}\left(\frac{1}{(n \xi^e_{b_{j-1}}(z))^2} \overset{e}{\mathfrak{M}}{}^\infty(z) f^e_{b_{j-1}}(n) (\overset{e}{\mathfrak{M}}{}^\infty(z))^{-1} \right), \qquad j = 1, \dots, N+1, \end{aligned}$$

where $\overset{e}{\mathfrak{M}}{}^\infty(z)$ *is characterised completely in Lemma* 4.5*,* $s_1 = 5/72$*,* $t_1 = -7/72$*, for* $j = 1, \dots, N+1$*,* $\xi^e_{b_{j-1}}(z) = -2 \int_z^{b^e_{j-1}} (R_e(s))^{1/2} h_V^e(s) \, \mathrm{d}s = (z - b^e_{j-1})^{3/2} G^e_{b_{j-1}}(z)$*, with* $G^e_{b_{j-1}}(z)$ *described completely in Lemma* 4.6*,* $\mho^e_{j-1}$ *is defined in Remark* 4.4*, and* $(f^e_{b_{j-1}}(n))_{kl} =_{n \to \infty} \mathcal{O}(1)$*,* $k, l = 1, 2$*, and*

$$\upsilon_{\mathcal{R}}^e(z) \underset{\substack{n \to \infty \\ z \in \mathbb{C}_\pm \cap \partial \mathbb{U}^e_{\delta_{a_j}}}}{=} \mathrm{I} + \frac{1}{n \xi^e_{a_j}(z)} \overset{e}{\mathfrak{M}}{}^\infty(z) \begin{pmatrix} \mp(s_1 + t_1) & \pm\mathrm{i}(s_1 - t_1)\mathrm{e}^{\mathrm{i}n\mho^e_j} \\ \pm\mathrm{i}(s_1 - t_1)\mathrm{e}^{-\mathrm{i}n\mho^e_j} & \pm(s_1 + t_1) \end{pmatrix} (\overset{e}{\mathfrak{M}}{}^\infty(z))^{-1}$$

$$+\mathcal{O}\left(\frac{1}{(n\xi^e_{a_j}(z))^2}\overset{e}{\mathfrak{M}}{}^{\infty}(z)f^e_{a_j}(n)(\overset{e}{\mathfrak{M}}{}^{\infty}(z))^{-1}\right),\qquad j=1,\dots,N+1,$$

where, for $j=1,\dots,N+1$, $\xi^e_{a_j}(z)=2\int_{a_j^e}^{z}(R_e(s))^{1/2}h^e_V(s)\,\mathrm{d}s=(z-a^e_j)^{3/2}G^e_{a_j}(z)$, *with* $G^e_{a_j}(z)$ *described completely in Lemma* 4.7, *and* $(f^e_{a_j}(n))_{kl}=_{n\to\infty}\mathcal{O}(1)$, $k,l=1,2$.

Proof. Recall the definition of $v^e_{\mathcal{R}}(z)$ given in Lemma 4.8. For $z\in\Sigma^{e,1}_p:=(-\infty,b^e_0-\delta^e_{b_0})\cup(a^e_{N+1}+\delta^e_{a_{N+1}},+\infty)$, recall from Lemma 4.8 that

$$v^e_{\mathcal{R}}(z):=v^{e,1}_{\mathcal{R}}(z)=\mathrm{I}+\exp\left(n(g^e_+(z)+g^e_-(z)-\widetilde{V}(z)-\ell_e+2Q_e)\right)\overset{e}{m}{}^{\infty}(z)\sigma_+(\overset{e}{m}{}^{\infty}(z))^{-1},$$

and, from the proof of Lemma 4.1, $g^e_+(z)+g^e_-(z)-\widetilde{V}(z)-\ell_e+2Q_e$ equals $-2\int_{a^e_{N+1}}^{z}(R_e(s))^{1/2}h^e_V(s)\,\mathrm{d}s$ (<0) for $z\in(a^e_{N+1}+\delta^e_{a_{N+1}},+\infty)$ and equals $2\int_z^{b^e_0}(R_e(s))^{1/2}h^e_V(s)\,\mathrm{d}s$ (<0) for $z\in(-\infty,b^e_0-\delta^e_{b_0})$; hence, recalling that $\widetilde{V}\colon\mathbb{R}\setminus\{0\}\to\mathbb{R}$, which is regular, satisfies conditions (2.3)–(2.5), using the asymptotic expansions (as $|z|\to\infty$ and $|z|\to 0$) for $g^e_+(z)+g^e_-(z)-\widetilde{V}(z)-\ell_e+2Q_e$ given in the proof of Lemma 3.6, that is, $g^e_+(z)+g^e_-(z)-\widetilde{V}(z)-\ell_e+2Q_e=_{|z|\to\infty}\ln(z^2+1)-\widetilde{V}(z)+\mathcal{O}(1)$ and $g^e_+(z)+g^e_-(z)-\widetilde{V}(z)-\ell_e+2Q_e=_{|z|\to 0}\ln(z^{-2}+1)-\widetilde{V}(z)+\mathcal{O}(1)$, upon recalling the expression for $\overset{e}{m}{}^{\infty}(z)$ given in Lemma 4.5 and noting that the respective factors $\gamma^e(z)\pm(\gamma^e(z))^{-1}$ and $\boldsymbol{\theta}^e(\pm\boldsymbol{u}^e(z)-\frac{n}{2\pi}\boldsymbol{\Omega}^e\pm\boldsymbol{d}_e)$ are uniformly bounded (with respect to z) in compact subsets outside the open intervals surrounding the end-points of the suppport of the 'even' equilibrium measure, defining $\mathbb{U}^e_0$ as in the Proposition, one arrives at the asymptotic (as $n\to\infty$) estimates for $v^e_{\mathcal{R}}(z)$ on $\Sigma^{e,1}_p\setminus\mathbb{U}^e_0\ni z$ and $\Sigma^{e,1}_p\cap\mathbb{U}^e_0\ni z$ stated in item (1) of the Proposition. (It should be noted that the n-dependence of the $\mathrm{GL}_2(\mathbb{C})$-valued factors $f_{\infty}(n)$ and $f_0(n)$ are inherited from the bounded ($\mathcal{O}(1)$) n-dependence of the respective Riemann theta functions, whose corresponding series converge absolutely and uniformly due to the fact that the associated Riemann matrix of $\boldsymbol{\beta}^e$-periods, τ^e, is pure imaginary and $-\mathrm{i}\tau^e$ is positive definite.)

For $z\in\Sigma^{e,2}_{p,j}:=(a^e_j+\delta^e_{a_j},b^e_j-\delta^e_{b_j})$, $j=1,\dots,N$, recall from Lemma 4.8 that

$$v^e_{\mathcal{R}}(z):=v^{e,2}_{\mathcal{R}}(z)=\mathrm{I}+\mathrm{e}^{-\mathrm{i}n\Omega^e_j}\exp\left(n(g^e_+(z)+g^e_-(z)-\widetilde{V}(z)-\ell_e+2Q_e)\right)\overset{e}{m}{}^{\infty}_-(z)\sigma_+(\overset{e}{m}{}^{\infty}_-(z))^{-1},$$

and, from the proof of Lemma 4.1, $g^e_+(z)+g^e_-(z)-\widetilde{V}(z)-\ell_e+2Q_e=-2\int_{a^e_j}^{z}(R_e(s))^{1/2}h^e_V(s)\,\mathrm{d}s$ (<0). Recalling, also, that $(R_e(z))^{1/2}:=(\prod_{k=1}^{N+1}(z-b^e_{k-1})(z-a^e_k))^{1/2}$ is continuous (and bounded) on the compact intervals $[a^e_j,b^e_j]\supset\Sigma^{e,2}_{p,j}\ni z$, $j=1,\dots,N$, vanishes at the end-points $\{a^e_j\}_{j=1}^N$ (resp., $\{b^e_j\}_{j=1}^N$) like $(R_e(z))^{1/2}=_{z\downarrow a^e_j}\mathcal{O}((z-a^e_j)^{1/2})$ (resp., $(R_e(z))^{1/2}=_{z\uparrow b^e_j}\mathcal{O}((b^e_j-z)^{1/2})$), and is differentiable on the open intervals $\Sigma^{e,2}_{p,j}\ni z$, and $h^e_V(z)=\frac{1}{2}\oint_{C^e_{\mathrm{R}}}(\frac{\mathrm{i}}{\pi s}+\frac{\mathrm{i}\widetilde{V}'(s)}{2\pi})(R_e(s))^{-1/2}(s-z)^{-1}\,\mathrm{d}s$ is analytic, it follows that, for $z\in\Sigma^{e,2}_{p,j}$,

$$\inf_{z\in\Sigma^{e,2}_{p,j}}(R_e(z))^{1/2}=:\widehat{m}_j\leqslant(R_e(z))^{1/2}\leqslant\widehat{M}_j:=\sup_{z\in\Sigma^{e,2}_{p,j}}(R_e(z))^{1/2},\qquad j=1,\dots,N;$$

thus, recalling the expression for $\overset{e}{m}{}^{\infty}(z)$ given in Lemma 4.5 and noting that the respective factors $\gamma^e(z)\pm(\gamma^e(z))^{-1}$ and $\boldsymbol{\theta}^e(\pm\boldsymbol{u}^e(z)-\frac{n}{2\pi}\boldsymbol{\Omega}^e\pm\boldsymbol{d}_e)$ are uniformly bounded (with respect to z) in compact subsets outside the open intervals surrounding the end-points of the suppport of the 'even' equilibrium measure, and defining $\mathbb{U}^e_0$ as in the Proposition, after a straightforward integration argument, one arrives at the asymptotic (as $n\to\infty$) estimates for $v^e_{\mathcal{R}}(z)$ on $\Sigma^{e,2}_{p,j}\setminus\mathbb{U}^e_0\ni z$ and $\Sigma^{e,2}_{p,j}\cap\mathbb{U}^e_0\ni z$, $j=1,\dots,N$, stated in item (2) of the Proposition (the n-dependence of the $\mathrm{GL}_2(\mathbb{C})$-valued factors $f_j(n),\widetilde{f}_j(n)$, $j=1,\dots,N$, is inherited from the bounded ($\mathcal{O}(1)$) n-dependence of the respective Riemann theta functions).

For $z\in\Sigma^{e,3}_p:=\cup_{j=1}^{N+1}(J^{e,\frown}_j\setminus(\mathbb{C}_+\cap(\mathbb{U}^e_{\delta_{b_{j-1}}}\cup\mathbb{U}^e_{\delta_{a_j}})))$, recall from the proof of Lemma 4.8 that

$$v^e_{\mathcal{R}}(z):=v^{e,3}_{\mathcal{R}}(z)=\mathrm{I}+\exp\left(-4n\pi\mathrm{i}\int_z^{a^e_{N+1}}\psi^e_V(s)\,\mathrm{d}s\right)\overset{e}{m}{}^{\infty}(z)\sigma_-(\overset{e}{m}{}^{\infty}(z))^{-1},$$

and, from Lemma 4.1, $\mathrm{Re}(\mathrm{i}\int_z^{a^e_{N+1}}\psi^e_V(s)\,\mathrm{d}s)>0$ for $z\in\mathbb{C}_+\cap(\cup_{j=1}^{N+1}\mathbb{U}^e_j)\supset\Sigma^{e,3}_p$, where $\mathbb{U}^e_j:=\{z\in\mathbb{C}^*;\ \mathrm{Re}(z)\in(b^e_{j-1},a^e_j),\ \inf_{q\in(b^e_{j-1},a^e_j)}|z-q|<r_j\in(0,1)\}$, $j=1,\dots,N+1$, with $\mathbb{U}^e_i\cap\mathbb{U}^e_j=\varnothing$, $i\neq j=1,\dots,N+1$: using

the expression for $\overset{e}{m}^{\infty}(z)$ given in Lemma 4.5 and noting that the respective factors $\gamma^{e}(z)\pm(\gamma^{e}(z))^{-1}$ and $\boldsymbol{\theta}^{e}(\pm\boldsymbol{u}^{e}(z)-\frac{n}{2\pi}\boldsymbol{\Omega}^{e}\pm\boldsymbol{d}_{e})$ are uniformly bounded (with respect to z) in compact subsets outside the open intervals surrounding the end-points of the suppport of the 'even' equilibrium measure, an arc-length-parametrisation argument, complemented by an application of the Maximum Length (ML) Theorem, leads one directly to the asymptotic (as $n\to\infty$) estimate for $\upsilon_{\mathcal{R}}^{e}(z)$ on $\Sigma_{p}^{e,3}\ni z$ stated in item (3) of the Proposition (the n-dependence of the $\mathrm{GL}_{2}(\mathbb{C})$-valued factor $\widehat{f}(n)$ is inherited from the bounded ($\mathcal{O}(1)$) n-dependence of the respective Riemann theta functions). The above argument applies, *mutatis mutandis*, for the asymptotic estimate of $\upsilon_{\mathcal{R}}^{e}(z)$ on $\Sigma_{p}^{e,4}:=\cup_{j=1}^{N+1}(J_{j}^{e,\smile}\setminus(\mathbb{C}_{-}\cap(\mathbb{U}_{\delta_{b_{j-1}}}^{e}\cup\mathbb{U}_{\delta_{a_{j}}}^{e})))\ni z$ stated in item (4) of the Proposition.

Since the estimates in item (5) of the Proposition are similar, consider, say, and without loss of generality, the asymptotic (as $n\to\infty$) estimate for $\upsilon_{\mathcal{R}}^{e}(z)$ on $\partial\mathbb{U}_{\delta_{a_{j}}}^{e}\ni z$, $j=1,\dots,N+1$: this argument applies, *mutatis mutandis*, for the large-n asymptotics of $\upsilon_{\mathcal{R}}^{e}(z)$ on $\cup_{j=1}^{N+1}\partial\mathbb{U}_{\delta_{b_{j-1}}}^{e}\ni z$. For $z\in\partial\mathbb{U}_{\delta_{a_{j}}}^{e}$, $j=1,\dots,N+1$, recall from the proof of Lemma 4.8 that $\upsilon_{\mathcal{R}}^{e}(z):=\upsilon_{\mathcal{R}}^{e,5}(z)=\mathcal{X}^{e}(z)(\overset{e}{m}^{\infty}(z))^{-1}$: using the expression for the parametrix, $\mathcal{X}^{e}(z)$, given in Lemma 4.7, and the large-argument asymptotics for the Airy function and its derivative given in Equations (2.6), one shows that, for $z\in\mathbb{C}_{+}\cap\partial\mathbb{U}_{\delta_{a_{j}}}^{e}$, $j=1,\dots,N+1$,

$$
\begin{aligned}
\upsilon_{\mathcal{R}}^{e}(z)\underset{n\to\infty}{=}\mathrm{I}+&\frac{\mathrm{e}^{-\frac{\mathrm{i}\pi}{3}}}{n\xi_{a_{j}}^{e}(z)}\overset{e}{m}^{\infty}(z)\begin{pmatrix}\mathrm{i}\mathrm{e}^{\frac{\mathrm{i}}{2}n\mho_{j}^{e}} & -\mathrm{i}\mathrm{e}^{\frac{\mathrm{i}}{2}n\mho_{j}^{e}}\\ \mathrm{e}^{-\frac{\mathrm{i}}{2}n\mho_{j}^{e}} & \mathrm{e}^{-\frac{\mathrm{i}}{2}n\mho_{j}^{e}}\end{pmatrix}\begin{pmatrix}-s_{1}\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}}\mathrm{e}^{-\frac{\mathrm{i}}{2}n\mho_{j}^{e}} & s_{1}\mathrm{e}^{\frac{\mathrm{i}\pi}{3}}\mathrm{e}^{\frac{\mathrm{i}}{2}n\mho_{j}^{e}}\\ t_{1}\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}}\mathrm{e}^{-\frac{\mathrm{i}}{2}n\mho_{j}^{e}} & -t_{1}\mathrm{e}^{\frac{4\pi\mathrm{i}}{3}}\mathrm{e}^{\frac{\mathrm{i}}{2}n\mho_{j}^{e}}\end{pmatrix}\\
&\times(\overset{e}{m}^{\infty}(z))^{-1}+\mathcal{O}\left(\frac{1}{(n\xi_{a_{j}}^{e}(z))^{2}}\overset{e}{m}^{\infty}(z)\begin{pmatrix}* & *\\ * & *\end{pmatrix}(\overset{e}{m}^{\infty}(z))^{-1}\right),
\end{aligned}
$$

where $\xi_{a_{j}}^{e}(z)$ and $\mho_{j}^{e}$, $j=1,\dots,N+1$, and s_{1} and t_{1} are defined in the Proposition, $\overset{e}{m}^{\infty}(z)$ is given in Lemma 4.5, and $\left(\begin{smallmatrix}* & *\\ * & *\end{smallmatrix}\right)\in\mathrm{M}_{2}(\mathbb{C})$, and, for $z\in\mathbb{C}_{-}\cap\partial\mathbb{U}_{\delta_{a_{j}}}^{e}$, $j=1,\dots,N+1$,

$$
\begin{aligned}
\upsilon_{\mathcal{R}}^{e}(z)\underset{n\to\infty}{=}\mathrm{I}+&\frac{\mathrm{e}^{-\frac{\mathrm{i}\pi}{3}}}{n\xi_{a_{j}}^{e}(z)}\overset{e}{m}^{\infty}(z)\begin{pmatrix}\mathrm{i}\mathrm{e}^{-\frac{\mathrm{i}}{2}n\mho_{j}^{e}} & -\mathrm{i}\mathrm{e}^{-\frac{\mathrm{i}}{2}n\mho_{j}^{e}}\\ \mathrm{e}^{\frac{\mathrm{i}}{2}n\mho_{j}^{e}} & \mathrm{e}^{\frac{\mathrm{i}}{2}n\mho_{j}^{e}}\end{pmatrix}\begin{pmatrix}-s_{1}\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}}\mathrm{e}^{\frac{\mathrm{i}}{2}n\mho_{j}^{e}} & s_{1}\mathrm{e}^{\frac{\mathrm{i}\pi}{3}}\mathrm{e}^{-\frac{\mathrm{i}}{2}n\mho_{j}^{e}}\\ t_{1}\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}}\mathrm{e}^{\frac{\mathrm{i}}{2}n\mho_{j}^{e}} & -t_{1}\mathrm{e}^{\frac{4\pi\mathrm{i}}{3}}\mathrm{e}^{-\frac{\mathrm{i}}{2}n\mho_{j}^{e}}\end{pmatrix}\\
&\times(\overset{e}{m}^{\infty}(z))^{-1}+\mathcal{O}\left(\frac{1}{(n\xi_{a_{j}}^{e}(z))^{2}}\overset{e}{m}^{\infty}(z)\begin{pmatrix}* & *\\ * & *\end{pmatrix}(\overset{e}{m}^{\infty}(z))^{-1}\right).
\end{aligned}
$$

Upon recalling the formula for $\overset{e}{m}^{\infty}(z)$ in terms of $\overset{e}{\mathfrak{M}}^{\infty}(z)$ given in Lemma 4.5, and noting that the respective factors $\gamma^{e}(z)\pm(\gamma^{e}(z))^{-1}$ and $\boldsymbol{\theta}^{e}(\pm\boldsymbol{u}^{e}(z)-\frac{n}{2\pi}\boldsymbol{\Omega}^{e}\pm\boldsymbol{d}_{e})$ are uniformly bounded (with respect to z) in compact subsets outside the open intervals surrounding the end-points of the suppport of the 'even' equilibrium measure, after a straightforward matrix-multiplication argument, one arrives at the asymptotic (as $n\to\infty$) estimates for $\upsilon_{\mathcal{R}}^{e}(z)$ on $\partial\mathbb{U}_{\delta_{a_{j}}}^{e}\ni z$, $j=1,\dots,N+1$, stated in item (5) of the Proposition (the n-dependence of the $\mathrm{GL}_{2}(\mathbb{C})$-valued factors $f_{a_{j}}^{e}(n)$, $j=1,\dots,N+1$, is inherited from the bounded ($\mathcal{O}(1)$) n-dependence of the respective Riemann theta functions). □

Definition 5.1. *For an oriented contour $D\subset\mathbb{C}$, let $\mathcal{N}_{q}(D)$ denote the set of all bounded linear operators from $\mathcal{L}_{\mathrm{M}_{2}(\mathbb{C})}^{q}(D)$ into $\mathcal{L}_{\mathrm{M}_{2}(\mathbb{C})}^{q}(D)$, $q\in\{1,2,\infty\}$.*

Since the analysis that follows relies substantially on the BC [84] construction for the solution of a matrix (and suitably normalised) RHP on an oriented and unbounded contour, it is convenient to present, with some requisite preamble, a succinct and self-contained synopsis of it at this juncture. One agrees to call a contour $\Gamma^{\sharp}$ *oriented* if:

(1) $\mathbb{C}\setminus\Gamma^{\sharp}$ has finitely many open connected components;
(2) $\mathbb{C}\setminus\Gamma^{\sharp}$ is the disjoint union of two, possibly disconnected, open regions, denoted by $\boldsymbol{\mho}^{+}$ and $\boldsymbol{\mho}^{-}$;
(3) $\Gamma^{\sharp}$ may be viewed as either the positively oriented boundary for $\boldsymbol{\mho}^{+}$ or the negatively oriented boundary for $\boldsymbol{\mho}^{-}$ ($\mathbb{C}\setminus\Gamma^{\sharp}$ is coloured by two colours, $\pm$).

Let $\Gamma^{\sharp}$, as a closed set, be the union of finitely many oriented, simple, piecewise-smooth arcs. Denote the set of all self-intersections of $\Gamma^{\sharp}$ by $\widehat{\Gamma}^{\sharp}$ (with $\mathrm{card}(\widehat{\Gamma}^{\sharp})<\infty$ assumed throughout). Set $\widetilde{\Gamma}^{\sharp}:=\Gamma^{\sharp}\setminus\widehat{\Gamma}^{\sharp}$.

The BC [84] construction for the solution of a (matrix) RHP, in the absence of a discrete spectrum and spectral singularities [101] (see, also, [85, 86, 102–104]), on an oriented contour $\Gamma^{\sharp}$ consists of finding function $\mathcal{Y}\colon \mathbb{C}\setminus\Gamma^{\sharp}\to \mathrm{M}_2(\mathbb{C})$ such that:

(1) $\mathcal{Y}(z)$ is holomorphic for $z\in\mathbb{C}\setminus\Gamma^{\sharp}$, $\mathcal{Y}\restriction_{\mathbb{C}\setminus\Gamma^{\sharp}}$ has a continuous extension (from 'above' and 'below') to $\widetilde{\Gamma}^{\sharp}$, and $\lim_{\substack{z'\to z\\ z'\,\in\pm\,\text{side of}\,\widetilde{\Gamma}^{\sharp}}}\int_{\widetilde{\Gamma}^{\sharp}}|\mathcal{Y}(z')-\mathcal{Y}_{\pm}(z)|^2\,|\mathrm{d}z|=0$;

(2) $\mathcal{Y}_{\pm}(z):=\lim_{\substack{z'\to z\\ z'\,\in\pm\,\text{side of}\,\widetilde{\Gamma}^{\sharp}}}\mathcal{Y}(z')$ satisfy $\mathcal{Y}_{+}(z)=\mathcal{Y}_{-}(z)v(z)$, $z\in\widetilde{\Gamma}^{\sharp}$, for some (smooth) 'jump' matrix $v\colon\widetilde{\Gamma}^{\sharp}\to\mathrm{GL}_2(\mathbb{C})$; and

(3) for arbitrarily fixed $\lambda_o\in\mathbb{C}$, and uniformly with respect to z, $\mathcal{Y}(z)\underset{\substack{z\to\lambda_o\\ z\in\mathbb{C}\setminus\Gamma^{\sharp}}}{=}\mathrm{I}+o(1)$, where $o(1)=\mathcal{O}(z-\lambda_o)$ if λ_o is finite, and $o(1)=\mathcal{O}(z^{-1})$ if λ_o is the point at infinity).

(Condition (3) is referred to as the *normalisation condition*, and is necessary in order to prove uniqueness of the associated RHP: one says that the RHP is 'normalised at λ_o'.) Let $v(z):=(\mathrm{I}-w_{-}(z))^{-1}(\mathrm{I}+w_{+}(z))$, $z\in\widetilde{\Gamma}^{\sharp}$, be a (bounded algebraic) factorisation for $v(z)$, where $w_{\pm}(z)$ are some upper/lower, or lower/upper, triangular matrices (depending on the orientation of $\Gamma^{\sharp}$), and $w_{\pm}(z)\in\cap_{p\in\{2,\infty\}}\mathcal{L}^{p}_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Gamma}^{\sharp})$ (if $\widetilde{\Gamma}^{\sharp}$ is unbounded, one requires that $w_{\pm}(z)\underset{\substack{z\to\infty\\ z\in\widetilde{\Gamma}^{\sharp}}}{=}\mathbf{0}$). Define $w(z):=w_{+}(z)+w_{-}(z)$, and introduce the (normalised at λ_o) Cauchy operators

$$\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Gamma^{\sharp})\ni f\mapsto(C^{\lambda_o}_{\pm}f)(z):=\lim_{\substack{z'\to z\\ z'\,\in\pm\,\text{side of}\,\Gamma^{\sharp}}}\int_{\Gamma^{\sharp}}\frac{(z'-\lambda_o)f(\zeta)}{(\zeta-\lambda_o)(\zeta-z')}\frac{\mathrm{d}\zeta}{2\pi\mathrm{i}},$$

where $\frac{(z-\lambda_o)}{(\zeta-\lambda_o)(\zeta-z)}\frac{\mathrm{d}\zeta}{2\pi\mathrm{i}}$ is the Cauchy kernel normalised at λ_o (which reduces to the 'standard' Cauchy kernel, that is, $\frac{1}{\zeta-z}\frac{\mathrm{d}\zeta}{2\pi\mathrm{i}}$, in the limit $\lambda_o\to\infty$), with $C^{\lambda_o}_{\pm}\colon\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Gamma^{\sharp})\to\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Gamma^{\sharp})$ bounded in operator norm[12], and $\|(C^{\lambda_o}_{\pm}f)(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Gamma^{\sharp})}\leqslant\text{const.}\|f(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Gamma^{\sharp})}$. Introduce the BC operator $C^{\lambda_o}_{w}$:

$$\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Gamma^{\sharp})\ni f\mapsto C^{\lambda_o}_{w}f:=C^{\lambda_o}_{+}(fw_{-})+C^{\lambda_o}_{-}(fw_{+}),$$

which, for $w_{\pm}\in\mathcal{L}^{\infty}_{\mathrm{M}_2(\mathbb{C})}(\Gamma^{\sharp})$, is bounded from $\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Gamma^{\sharp})\to\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Gamma^{\sharp})$, that is, $\|C^{\lambda_o}_{w}\|_{\mathcal{N}_2(\Gamma^{\sharp})}<\infty$; furthermore, since $\mathbb{C}\setminus\Gamma^{\sharp}$ can be coloured by the two colours $\pm$, $C^{\lambda_o}_{\pm}$ are complementary projections [2, 85, 102, 103], that is, $(C^{\lambda_o}_{+})^2=C^{\lambda_o}_{+}$, $(C^{\lambda_o}_{-})^2=-C^{\lambda_o}_{-}$, $C^{\lambda_o}_{+}C^{\lambda_o}_{-}=C^{\lambda_o}_{-}C^{\lambda_o}_{+}=\underline{\mathbf{0}}$ (the null operator), and $C^{\lambda_o}_{+}-C^{\lambda_o}_{-}=\mathbf{id}$ (the identity operator). (In the case that $C^{\lambda_o}_{+}$ and $-C^{\lambda_o}_{-}$ are complementary, the contour $\Gamma^{\sharp}$ can always be oriented in such a way that the $\pm$ regions lie on the $\pm$ sides of the contour, respectively.) The solution of the above (normalised at λ_o) RHP is given by the following integral representation.

Lemma 5.1 (Beals and Coifman [84]**).** *Set*

$$\mu_{\lambda_o}(z)=\mathcal{Y}_{+}(z)(\mathrm{I}+w_{+}(z))^{-1}=\mathcal{Y}_{-}(z)(\mathrm{I}-w_{-}(z))^{-1},\quad z\in\Gamma^{\sharp}.$$

If $\mu_{\lambda_o}\in\mathrm{I}+\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Gamma^{\sharp})$ *solves the linear singular integral equation*

$$\left(\mathbf{id}-C^{\lambda_o}_{w}\right)(\mu_{\lambda_o}(z)-\mathrm{I})=C^{\lambda_o}_{w}\mathrm{I}=C^{\lambda_o}_{+}(w_{-}(z))+C^{\lambda_o}_{-}(w_{+}(z)),\quad z\in\Gamma^{\sharp},$$

where $\mathbf{id}$ *is the identity operator on* $\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Gamma^{\sharp})$, *then the solution of the* RHP $(\mathcal{Y}(z),v(z),\Gamma^{\sharp})$ *is given by*

$$\mathcal{Y}(z)=\mathrm{I}+\int_{\Gamma^{\sharp}}\frac{(z-\lambda_o)\mu_{\lambda_o}(\zeta)w(\zeta)}{(\zeta-\lambda_o)(\zeta-z)}\frac{\mathrm{d}\zeta}{2\pi\mathrm{i}},\quad z\in\mathbb{C}\setminus\Gamma^{\sharp},$$

where $\mu_{\lambda_o}(z):=((\mathbf{id}-C^{\lambda_o}_{w})^{-1}\mathrm{I})(z)$[13].

[12] $\|C^{\lambda_o}_{\pm}\|_{\mathcal{N}_2(\Gamma^{\sharp})}<\infty$.

[13] The linear singular integral equation for $\mu_{\lambda_o}(\cdot)$ stated in this Lemma 5.1 is well defined in $\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Gamma^{\sharp})$ provided that $w_{\pm}(\cdot)\in\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Gamma^{\sharp})\cap\mathcal{L}^{\infty}_{\mathrm{M}_2(\mathbb{C})}(\Gamma^{\sharp})$; furthermore, it is assumed that the associated RHP $(\mathcal{Y}(z),v(z),\Gamma^{\sharp})$ is solvable, that is, $\dim\ker(\mathbf{id}-C^{\lambda_o}_{w})=\dim\left\{\phi\in\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Gamma^{\sharp});\ (\mathbf{id}-C^{\lambda_o}_{w})\phi=\underline{\mathbf{0}}\right\}=\dim\varnothing=0$ ($\Rightarrow(\mathbf{id}-C^{\lambda_o}_{w})^{-1}\restriction_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Gamma^{\sharp})}$ exists).

Recall that $\mathcal{R}^e\colon \mathbb{C}\setminus\Sigma_p^e \to \mathrm{SL}_2(\mathbb{C})$, which solves the RHP $(\mathcal{R}^e(z), v^e_{\mathcal{R}}(z), \Sigma_p^e)$ formulated in Lemma 4.8, is normalised at infinity, that is, $\mathcal{R}^e(\infty)=\mathrm{I}$. Removing from the specification of the RHP $(\mathcal{R}^e(z), v^e_{\mathcal{R}}(z), \Sigma_p^e)$ the oriented skeletons on which the jump matrix, $v^e_{\mathcal{R}}(z)$, is equal to I, in particular (cf. Lemma 4.8), the oriented skeleton $\Sigma_p^e \setminus \cup_{l=1}^{5} \Sigma_p^{e,l}$, and setting $\Sigma_p^e \setminus (\Sigma_p^e \setminus \cup_{l=1}^{5}\Sigma_p^{e,l}) =: \widetilde{\Sigma}_p^e$ (see Figure 10), one

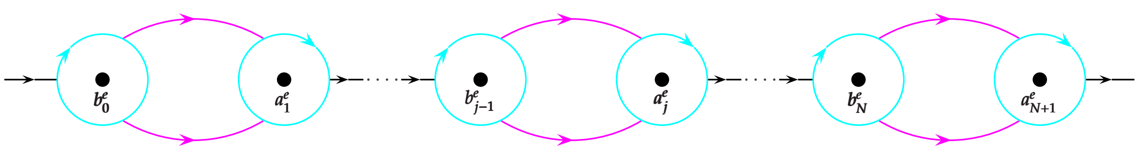

Figure 10: Oriented skeleton $\widetilde{\Sigma}_p^e := \Sigma_p^e \setminus (\Sigma_p^e \setminus \cup_{l=1}^{5}\Sigma_p^{e,l})$

arrives at the equivalent RHP $(\mathcal{R}^e(z), v^e_{\mathcal{R}}(z), \widetilde{\Sigma}_p^e)$ for $\mathcal{R}^e\colon \mathbb{C}\setminus\widetilde{\Sigma}_p^e \to \mathrm{SL}_2(\mathbb{C})$ (the normalisation at infinity, of course, remains unchanged). Via the BC [84] construction discussed above, write, for $v^e_{\mathcal{R}}\colon \widetilde{\Sigma}_p^e \to \mathrm{SL}_2(\mathbb{C})$, the (bounded algebraic) factorisation

$$v^e_{\mathcal{R}}(z) := \left(\mathrm{I} - w_-^{\Sigma^e_{\mathcal{R}}}(z)\right)^{-1}\left(\mathrm{I} + w_+^{\Sigma^e_{\mathcal{R}}}(z)\right), \quad z \in \widetilde{\Sigma}_p^e :$$

taking the (so-called) trivial factorisation [86] (see pp. 293 and 294, *Proof of Theorem* 3.14 and *Proposition* 1.9; see, also, [103, 104]) $w_-^{\Sigma^e_{\mathcal{R}}}(z) \equiv \mathbf{0}$, whence $v^e_{\mathcal{R}}(z) = \mathrm{I} + w_+^{\Sigma^e_{\mathcal{R}}}(z)$, $z \in \widetilde{\Sigma}_p^e$, it follows from Lemma 5.1 that, upon normalising the Cauchy (integral) operator(s) at infinity (take the limit $\lambda_o \to \infty$ in Lemma 5.1), the ($\mathrm{SL}_2(\mathbb{C})$-valued) integral representation for the—unique—solution of the equivalent RHP $(\mathcal{R}^e(z), v^e_{\mathcal{R}}(z), \widetilde{\Sigma}_p^e)$ is

$$\mathcal{R}^e(z) = \mathrm{I} + \int_{\widetilde{\Sigma}_p^e} \frac{\mu^{\Sigma^e_{\mathcal{R}}}(s) w_+^{\Sigma^e_{\mathcal{R}}}(s)}{s-z}\,\frac{\mathrm{d}s}{2\pi\mathrm{i}}, \quad z \in \mathbb{C}\setminus\widetilde{\Sigma}_p^e, \tag{5.1}$$

where $\mu^{\Sigma^e_{\mathcal{R}}}(\cdot) \in \mathrm{I} + \mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}_p^e)$ solves the (linear) singular integral equation

$$(\mathbf{id} - C^{\infty}_{w^{\Sigma^e_{\mathcal{R}}}})\mu^{\Sigma^e_{\mathcal{R}}}(z) = \mathrm{I}, \quad z \in \widetilde{\Sigma}_p^e,$$

with

$$\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}_p^e) \ni f \mapsto C^{\infty}_{w^{\Sigma^e_{\mathcal{R}}}} f := C^{\infty}_{-}(f w_+^{\Sigma^e_{\mathcal{R}}}),$$

and

$$\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}_p^e) \ni f \mapsto (C^{\infty}_{\pm} f)(z) := \lim_{\substack{z' \to z \\ z' \in \pm\,\text{side of}\,\widetilde{\Sigma}_p^e}} \int_{\widetilde{\Sigma}_p^e} \frac{f(s)}{s-z'}\,\frac{\mathrm{d}s}{2\pi\mathrm{i}};$$

furthermore, $\|C^{\infty}_{\pm}\|_{\mathcal{N}_2(\widetilde{\Sigma}_p^e)} < \infty$.

Proposition 5.2. *Let* $\mathcal{R}^e\colon \mathbb{C}\setminus\widetilde{\Sigma}_p^e \to \mathrm{SL}_2(\mathbb{C})$ *solve the following, equivalent* RHP: (i) $\mathcal{R}^e(z)$ *is holomorphic for* $z \in \mathbb{C}\setminus\widetilde{\Sigma}_p^e$; (ii) $\mathcal{R}^e_{\pm}(z) := \lim_{\substack{z' \to z \\ z' \in \pm\,\text{side of}\,\widetilde{\Sigma}_p^e}} \mathcal{R}^e(z')$ *satisfy the boundary condition*

$$\mathcal{R}^e_+(z) = \mathcal{R}^e_-(z) v^e_{\mathcal{R}}(z), \quad z \in \widetilde{\Sigma}_p^e,$$

where $v^e_{\mathcal{R}}(z)$, *for* $z \in \widetilde{\Sigma}_p^e$, *is defined in Lemma* 4.8 *and satisfies the asymptotic (as* $n \to \infty$*) estimates given in Proposition* 5.1; (iii) $\mathcal{R}^e(z) \underset{\substack{z\to\infty \\ z\in\mathbb{C}\setminus\widetilde{\Sigma}_p^e}}{=} \mathrm{I} + \mathcal{O}(z^{-1})$; *and* (iv) $\mathcal{R}^e(z) \underset{\substack{z\to 0 \\ z\in\mathbb{C}\setminus\widetilde{\Sigma}_p^e}}{=} \mathcal{O}(1)$. *Then:*

(1) *for* $z \in (-\infty, b_0^e - \delta^e_{b_0}) \cup (a^e_{N+1} + \delta^e_{a_{N+1}}, +\infty) =: \Sigma_p^{e,1}$,

$$\|w_+^{\Sigma^e_{\mathcal{R}}}(\cdot)\|_{\mathcal{L}^q_{\mathrm{M}_2(\mathbb{C})}(\Sigma_p^{e,1})} \underset{n\to\infty}{=} \mathcal{O}\!\left(\frac{f(n)\mathrm{e}^{-nc}}{n^{1/q}}\right), \quad q = 1, 2, \qquad \|w_+^{\Sigma^e_{\mathcal{R}}}(\cdot)\|_{\mathcal{L}^\infty_{\mathrm{M}_2(\mathbb{C})}(\Sigma_p^{e,1})} \underset{n\to\infty}{=} \mathcal{O}(f(n)\mathrm{e}^{-nc}),$$

where $c>0$ *and* $f(n)=_{n\to\infty}\mathcal{O}(1)$;

(2) *for* $z\in(a_j^e+\delta_{a_j}^e, b_j^e-\delta_{b_j}^e)=:\Sigma_{p,j}^{e,2}\subset\cup_{l=1}^N\Sigma_{p,l}^{e,2}=:\Sigma_p^{e,2}$, $j=1,\ldots,N$,

$$\|w_+^{\Sigma_{\mathcal{R}}^e}(\cdot)\|_{\mathcal{L}^q_{\mathrm{M}_2(\mathbb{C})}(\Sigma_{p,j}^{e,2})}\underset{n\to\infty}{=}\mathcal{O}\left(\frac{f_j(n)\mathrm{e}^{-nc_j}}{n^{1/q}}\right),\quad q=1,2,\quad \|w_+^{\Sigma_{\mathcal{R}}^e}(\cdot)\|_{\mathcal{L}^\infty_{\mathrm{M}_2(\mathbb{C})}(\Sigma_{p,j}^{e,2})}\underset{n\to\infty}{=}\mathcal{O}\Big(f_j(n)\mathrm{e}^{-nc_j}\Big),$$

where $c_j>0$ *and* $f_j(n)=_{n\to\infty}\mathcal{O}(1)$;

(3) *for* $z\in\cup_{j=1}^{N+1}(J_j^{e,\frown}\setminus(\mathbb{C}_+\cap(\mathbb{U}^e_{\delta_{b_{j-1}}}\cup\mathbb{U}^e_{\delta_{a_j}})))=:\Sigma_p^{e,3}$,

$$\|w_+^{\Sigma_{\mathcal{R}}^e}(\cdot)\|_{\mathcal{L}^q_{\mathrm{M}_2(\mathbb{C})}(\Sigma_p^{e,3})}\underset{n\to\infty}{=}\mathcal{O}\left(\frac{f(n)\mathrm{e}^{-nc}}{n^{1/q}}\right),\quad q=1,2,\qquad \|w_+^{\Sigma_{\mathcal{R}}^e}(\cdot)\|_{\mathcal{L}^\infty_{\mathrm{M}_2(\mathbb{C})}(\Sigma_p^{e,3})}\underset{n\to\infty}{=}\mathcal{O}(f(n)\mathrm{e}^{-nc}),$$

where $c>0$ *and* $f(n)=_{n\to\infty}\mathcal{O}(1)$;

(4) *for* $z\in\cup_{j=1}^{N+1}(J_j^{e,\smile}\setminus(\mathbb{C}_-\cap(\mathbb{U}^e_{\delta_{b_{j-1}}}\cup\mathbb{U}^e_{\delta_{a_j}})))=:\Sigma_p^{e,4}$,

$$\|w_+^{\Sigma_{\mathcal{R}}^e}(\cdot)\|_{\mathcal{L}^q_{\mathrm{M}_2(\mathbb{C})}(\Sigma_p^{e,4})}\underset{n\to\infty}{=}\mathcal{O}\left(\frac{f(n)\mathrm{e}^{-nc}}{n^{1/q}}\right),\quad q=1,2,\qquad \|w_+^{\Sigma_{\mathcal{R}}^e}(\cdot)\|_{\mathcal{L}^\infty_{\mathrm{M}_2(\mathbb{C})}(\Sigma_p^{e,4})}\underset{n\to\infty}{=}\mathcal{O}(f(n)\mathrm{e}^{-nc}),$$

where $c>0$ *and* $f(n)=_{n\to\infty}\mathcal{O}(1)$; *and*

(5) *for* $z\in\cup_{j=1}^{N+1}(\partial\mathbb{U}^e_{\delta_{b_{j-1}}}\cup\partial\mathbb{U}^e_{\delta_{a_j}})=:\Sigma_p^{e,5}$,

$$\|w_+^{\Sigma_{\mathcal{R}}^e}(\cdot)\|_{\mathcal{L}^q_{\mathrm{M}_2(\mathbb{C})}(\Sigma_p^{e,5})}\underset{n\to\infty}{=}\mathcal{O}\Big(n^{-1}f(n)\Big),\quad q\in\{1,2,\infty\},$$

where $f(n)=_{n\to\infty}\mathcal{O}(1)$.

Furthermore,

$$\|C^\infty_{w^{\Sigma_{\mathcal{R}}^e}}\|_{\mathcal{N}_r(\widetilde{\Sigma}_p^e)}\underset{n\to\infty}{=}\mathcal{O}\Big(n^{-1}f(n)\Big),\quad r\in\{2,\infty\},$$

where $f(n)=_{n\to\infty}\mathcal{O}(1)$; *in particular,* $(\mathbf{id}-C^\infty_{w^{\Sigma_{\mathcal{R}}^e}})^{-1}\restriction_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}_p^e)}$ *exists, that is,*

$$\|(\mathbf{id}-C^\infty_{w^{\Sigma_{\mathcal{R}}^e}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}_p^e)}\underset{n\to\infty}{=}\mathcal{O}(1),$$

and it can be expanded in a Neumann series.

Proof. Without loss of generality, assume that $0\in\Sigma_p^{e,1}$ (cf. Proposition 5.1). Recall that $w_+^{\Sigma_{\mathcal{R}}^e}(z)=v_{\mathcal{R}}^e(z)-\mathrm{I}$, $z\in\widetilde{\Sigma}_p^e$. For $z\in\Sigma_p^{e,1}$, using the asymptotic (as $n\to\infty$) estimate for $v_{\mathcal{R}}^e(z)$ given in item (1) of Proposition 5.1, one gets that

$$\begin{aligned}
\|w_+^{\Sigma_{\mathcal{R}}^e}(\cdot)\|_{\mathcal{L}^\infty_{\mathrm{M}_2(\mathbb{C})}(\Sigma_p^{e,1})}&:=\max_{i,j=1,2}\sup_{z\in\Sigma_p^{e,1}}|(w_+^{\Sigma_{\mathcal{R}}^e}(z))_{ij}|\underset{n\to\infty}{=}\mathcal{O}(f(n)\mathrm{e}^{-nc}),\\
\|w_+^{\Sigma_{\mathcal{R}}^e}(\cdot)\|_{\mathcal{L}^1_{\mathrm{M}_2(\mathbb{C})}(\Sigma_p^{e,1})}&:=\int_{\Sigma_p^{e,1}}|w_+^{\Sigma_{\mathcal{R}}^e}(z)|\,|\mathrm{d}z|=\int_{(\Sigma_p^{e,1}\setminus\mathbb{U}_0^e)\cup\mathbb{U}_0^e}|w_+^{\Sigma_{\mathcal{R}}^e}(z)|\,|\mathrm{d}z|\\
&=\left(\int_{\Sigma_p^{e,1}\setminus\mathbb{U}_0^e}+\int_{\mathbb{U}_0^e}\right)\left(\sum_{i,j=1}^2(w_+^{\Sigma_{\mathcal{R}}^e}(z))_{ij}\overline{(w_+^{\Sigma_{\mathcal{R}}^e}(z))_{ij}}\right)^{1/2}|\mathrm{d}z|\\
&\underset{n\to\infty}{=}\mathcal{O}(n^{-1}f(n)\mathrm{e}^{-nc})+\mathcal{O}(n^{-1}f(n)\mathrm{e}^{-nc})\underset{n\to\infty}{=}\mathcal{O}(n^{-1}f(n)\mathrm{e}^{-nc})
\end{aligned}$$

($|\mathrm{d}z|$ denotes arc length), and

$$\begin{aligned}
\|w_+^{\Sigma_{\mathcal{R}}^e}(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Sigma_p^{e,1})}&:=\left(\int_{\Sigma_p^{e,1}}|w_+^{\Sigma_{\mathcal{R}}^e}(z)|^2\,|\mathrm{d}z|\right)^{1/2}=\left(\int_{(\Sigma_p^{e,1}\setminus\mathbb{U}_0^e)\cup\mathbb{U}_0^e}|w_+^{\Sigma_{\mathcal{R}}^e}(z)|^2\,|\mathrm{d}z|\right)^{1/2}\\
&=\left(\left(\int_{\Sigma_p^{e,1}\setminus\mathbb{U}_0^e}+\int_{\mathbb{U}_0^e}\right)\left(\sum_{i,j=1}^2(w_+^{\Sigma_{\mathcal{R}}^e}(z))_{ij}\overline{(w_+^{\Sigma_{\mathcal{R}}^e}(z))_{ij}}\right)|\mathrm{d}z|\right)^{1/2}
\end{aligned}$$

$$\underset{n\to\infty}{=}\Big(\mathcal{O}(n^{-1}f(n)\mathrm{e}^{-nc})+\mathcal{O}(n^{-1}f(n)\mathrm{e}^{-nc})\Big)^{1/2}\underset{n\to\infty}{=}\mathcal{O}(n^{-1/2}f(n)\mathrm{e}^{-nc}),$$

where $c>0$ and $f(n)=_{n\to\infty}\mathcal{O}(1)$.

For $z\in(a_j^e+\delta_{a_j}^e,b_j^e-\delta_{b_j}^e)=:\Sigma_{p,j}^{e,2}\subset\cup_{l=1}^N\Sigma_{p,l}^{e,2}=:\Sigma_p^{e,2}$ $(\subset\widetilde{\Sigma}_p^e)$, $j=1,\dots,N$, using the asymptotic (as $n\to\infty$) estimate for $v_{\mathcal{R}}^e(z)$ given in item (2) of Proposition 5.1, one gets that

$$\|w_+^{\Sigma_{\mathcal{R}}^e}(\cdot)\|_{\mathcal{L}^\infty_{\mathrm{M}_2(\mathbb{C})}(\Sigma_{p,j}^{e,2})}:=\max_{k,m=1,2}\sup_{z\in\Sigma_{p,j}^{e,2}}|(w_+^{\Sigma_{\mathcal{R}}^e}(z))_{km}|\underset{n\to\infty}{=}\mathcal{O}\Big(f_j(n)\mathrm{e}^{-nc_j}\Big),$$

$$\begin{aligned}\|w_+^{\Sigma_{\mathcal{R}}^e}(\cdot)\|_{\mathcal{L}^1_{\mathrm{M}_2(\mathbb{C})}(\Sigma_{p,j}^{e,2})}&:=\int_{\Sigma_{p,j}^{e,2}}|w_+^{\Sigma_{\mathcal{R}}^e}(z)|\,|\mathrm{d}z|=\int_{\Sigma_{p,j}^{e,2}}\left(\sum_{i,j=1}^2(w_+^{\Sigma_{\mathcal{R}}^e}(z))_{ij}\overline{(w_+^{\Sigma_{\mathcal{R}}^e}(z))_{ij}}\right)^{1/2}|\mathrm{d}z|\\&\underset{n\to\infty}{=}\mathcal{O}(n^{-1}f_j(n)\mathrm{e}^{-nc_j}),\end{aligned}$$

and

$$\begin{aligned}\|w_+^{\Sigma_{\mathcal{R}}^e}(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Sigma_{p,j}^{e,2})}&:=\left(\int_{\Sigma_{p,j}^{e,2}}|w_+^{\Sigma_{\mathcal{R}}^e}(z)|^2\,|\mathrm{d}z|\right)^{1/2}=\left(\int_{\Sigma_{p,j}^{e,2}}\sum_{k,l=1}^2(w_+^{\Sigma_{\mathcal{R}}^e}(z))_{kl}\overline{(w_+^{\Sigma_{\mathcal{R}}^e}(z))_{kl}}\,|\mathrm{d}z|\right)^{1/2}\\&\underset{n\to\infty}{=}\mathcal{O}(n^{-1/2}f_j(n)\mathrm{e}^{-nc_j}),\end{aligned}$$

where $c_j>0$ and $f_j(n)=_{n\to\infty}\mathcal{O}(1)$, $j=1,\dots,N$.

For $z\in\cup_{j=1}^{N+1}(J_j^{e,\frown}\setminus(\mathbb{C}_+\cap(\mathbb{U}_{\delta_{b_{j-1}}}^e\cup\mathbb{U}_{\delta_{a_j}}^e)))=:\Sigma_p^{e,3}$ $(\subset\widetilde{\Sigma}_p^e)$, using the asymptotic (as $n\to\infty$) estimate for $v_{\mathcal{R}}^e(z)$ given in item (3) of Proposition 5.1, one gets that

$$\|w_+^{\Sigma_{\mathcal{R}}^e}(\cdot)\|_{\mathcal{L}^\infty_{\mathrm{M}_2(\mathbb{C})}(\Sigma_p^{e,3})}:=\max_{i,j=1,2}\sup_{z\in\Sigma_p^{e,3}}|(w_+^{\Sigma_{\mathcal{R}}^e}(z))_{ij}|\underset{n\to\infty}{=}\mathcal{O}(f(n)\mathrm{e}^{-nc}),$$

$$\begin{aligned}\|w_+^{\Sigma_{\mathcal{R}}^e}(\cdot)\|_{\mathcal{L}^1_{\mathrm{M}_2(\mathbb{C})}(\Sigma_p^{e,3})}&:=\int_{\Sigma_p^{e,3}}|w_+^{\Sigma_{\mathcal{R}}^e}(z)|\,|\mathrm{d}z|=\int_{\Sigma_p^{e,3}}\left(\sum_{i,j=1}^2(w_+^{\Sigma_{\mathcal{R}}^e}(z))_{ij}\overline{(w_+^{\Sigma_{\mathcal{R}}^e}(z))_{ij}}\right)^{1/2}|\mathrm{d}z|\\&\underset{n\to\infty}{=}\mathcal{O}(n^{-1}f(n)\mathrm{e}^{-nc}),\end{aligned}$$

and

$$\begin{aligned}\|w_+^{\Sigma_{\mathcal{R}}^e}(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Sigma_p^{e,3})}&:=\left(\int_{\Sigma_p^{e,3}}|w_+^{\Sigma_{\mathcal{R}}^e}(z)|^2\,|\mathrm{d}z|\right)^{1/2}=\left(\int_{\Sigma_p^{e,3}}\sum_{i,j=1}^2(w_+^{\Sigma_{\mathcal{R}}^e}(z))_{ij}\overline{(w_+^{\Sigma_{\mathcal{R}}^e}(z))_{ij}}\,|\mathrm{d}z|\right)^{1/2}\\&\underset{n\to\infty}{=}\mathcal{O}(n^{-1/2}f(n)\mathrm{e}^{-nc}),\end{aligned}$$

where $c>0$ and $f(n)=_{n\to\infty}\mathcal{O}(1)$: the above analysis applies, *mutatis mutandis*, for the analogous estimates on $\Sigma_p^{e,4}:=\cup_{j=1}^{N+1}(J_j^{e,\smile}\setminus(\mathbb{C}_-\cap(\mathbb{U}_{\delta_{b_{j-1}}}^e\cup\mathbb{U}_{\delta_{a_j}}^e)))\ni z$.

For $z\in\cup_{j=1}^{N+1}(\partial\mathbb{U}_{\delta_{b_{j-1}}}^e\cup\partial\mathbb{U}_{\delta_{a_j}}^e)=:\Sigma_p^{e,5}$ $(\subset\widetilde{\Sigma}_p^e)$, using the $(2(N+1))$ asymptotic (as $n\to\infty$) estimates for $v_{\mathcal{R}}^e(z)$ given in item (5) of Proposition 5.1, one gets that

$$\|w_+^{\Sigma_{\mathcal{R}}^e}(\cdot)\|_{\mathcal{L}^\infty_{\mathrm{M}_2(\mathbb{C})}(\Sigma_p^{e,5})}:=\max_{i,j=1,2}\sup_{z\in\Sigma_p^{e,5}}|(w_+^{\Sigma_{\mathcal{R}}^e}(z))_{ij}|\underset{n\to\infty}{=}\mathcal{O}\Big(n^{-1}f(n)\Big),$$

$$\begin{aligned}\|w_+^{\Sigma_{\mathcal{R}}^e}(\cdot)\|_{\mathcal{L}^1_{\mathrm{M}_2(\mathbb{C})}(\Sigma_p^{e,5})}&:=\int_{\Sigma_p^{e,5}}|w_+^{\Sigma_{\mathcal{R}}^e}(z)|\,|\mathrm{d}z|=\int_{\cup_{j=1}^{N+1}(\partial\mathbb{U}_{\delta_{b_{j-1}}}^e\cup\partial\mathbb{U}_{\delta_{a_j}}^e)}|w_+^{\Sigma_{\mathcal{R}}^e}(z)|\,|\mathrm{d}z|\\&=\sum_{k=1}^{N+1}\left(\int_{\partial\mathbb{U}_{\delta_{b_{k-1}}}^e}+\int_{\partial\mathbb{U}_{\delta_{a_k}}^e}\right)\left(\sum_{i,j=1}^2(w_+^{\Sigma_{\mathcal{R}}^e}(z))_{ij}\overline{(w_+^{\Sigma_{\mathcal{R}}^e}(z))_{ij}}\right)^{1/2}|\mathrm{d}z|,\end{aligned}$$

whence, (cf. Lemma 4.5) using the fact that the respective factors $\gamma^{e}(z)\pm(\gamma^{e}(z))^{-1}$ and $\boldsymbol{\theta}^{e}(\pm\boldsymbol{u}^{e}(z)-\frac{n}{2\pi}\boldsymbol{\Omega}^{e}\pm\boldsymbol{d}_{e})$ are uniformly bounded (with respect to z) in compact intervals outside open intervals surrounding the end-points of the support of the 'even' equilibrium measure, one arrives at

$$\begin{aligned}
\|w_{+}^{\Sigma_{\mathcal{R}}^{e}}(\cdot)\|_{\mathcal{L}^{1}_{\mathrm{M}_2(\mathbb{C})}(\Sigma_p^{e,5})} &\underset{n\to\infty}{=} \frac{1}{n}\sum_{k=1}^{N+1}\left(\int_{\partial\mathbb{U}^{e}_{\delta_{b_{k-1}}}}\frac{|\star^{e}_{b_{k-1}}(z;n)|}{(z-b^{e}_{k-1})^{3/2}}\,|\mathrm{d}z|+\int_{\partial\mathbb{U}^{e}_{\delta_{a_k}}}\frac{|\star^{e}_{a_k}(z;n)|}{(z-a^{e}_{k})^{3/2}}\,|\mathrm{d}z|\right)\\
&\underset{n\to\infty}{=} \mathcal{O}(n^{-1}f(n)),
\end{aligned}$$

and, similarly,

$$\begin{aligned}
\|w_{+}^{\Sigma_{\mathcal{R}}^{e}}(\cdot)\|_{\mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\Sigma_p^{e,5})} &:= \left(\int_{\Sigma_p^{e,5}}|w_{+}^{\Sigma_{\mathcal{R}}^{e}}(z)|^{2}\,|\mathrm{d}z|\right)^{1/2}=\left(\int_{\cup_{j=1}^{N+1}(\partial\mathbb{U}^{e}_{\delta_{b_{j-1}}}\cup\partial\mathbb{U}^{e}_{\delta_{a_j}})}|w_{+}^{\Sigma_{\mathcal{R}}^{e}}(z)|^{2}\,|\mathrm{d}z|\right)^{1/2}\\
&= \left(\sum_{k=1}^{N+1}\left(\int_{\partial\mathbb{U}^{e}_{\delta_{b_{k-1}}}}+\int_{\partial\mathbb{U}^{e}_{\delta_{a_k}}}\right)\sum_{i,j=1}^{2}(w_{+}^{\Sigma_{\mathcal{R}}^{e}}(z))_{ij}\overline{(w_{+}^{\Sigma_{\mathcal{R}}^{e}}(z))_{ij}}\,|\mathrm{d}z|\right)^{1/2}\\
&\underset{n\to\infty}{=} \left(\frac{1}{n^{2}}\sum_{k=1}^{N+1}\left(\int_{\partial\mathbb{U}^{e}_{\delta_{b_{k-1}}}}\frac{|\star^{e}_{b_{k-1}}(z;n)|}{(z-b^{e}_{k-1})^{3}}\,|\mathrm{d}z|+\int_{\partial\mathbb{U}^{e}_{\delta_{a_k}}}\frac{|\star^{e}_{a_k}(z;n)|}{(z-a^{e}_{k})^{3}}\,|\mathrm{d}z|\right)\right)^{1/2}\\
&\underset{n\to\infty}{=} \mathcal{O}(n^{-1}f(n)),
\end{aligned}$$

where $f(n)=_{n\to\infty}\mathcal{O}(1)$.

Recall that $C^{\infty}_{w^{\Sigma_{\mathcal{R}}^{e}}}f:=C^{\infty}_{-}(fw_{+}^{\Sigma_{\mathcal{R}}^{e}})$, where $(C^{\infty}_{-}g)(z):=\lim_{\substack{z'\to z\\ z'\in-\widetilde{\Sigma}_p^{e}}}\int_{\widetilde{\Sigma}_p^{e}}\frac{g(s)}{s-z'}\frac{\mathrm{d}s}{2\pi\mathrm{i}}$, with $-\widetilde{\Sigma}_p^{e}$ shorthand for 'the $-$ side of $\widetilde{\Sigma}_p^{e}$'. For the $\|C^{\infty}_{w^{\Sigma_{\mathcal{R}}^{e}}}\|_{\mathcal{N}_{\infty}(\widetilde{\Sigma}_p^{e})}$ norm, one proceeds as follows:

$$\begin{aligned}
\|C^{\infty}_{w^{\Sigma_{\mathcal{R}}^{e}}}g\|_{\mathcal{L}^{\infty}_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}_p^{e})} &:= \max_{j,l=1,2}\sup_{z\in\widetilde{\Sigma}_p^{e}}|(C^{\infty}_{w^{\Sigma_{\mathcal{R}}^{e}}}g)_{jl}(z)|=\max_{j,l=1,2}\sup_{z\in\widetilde{\Sigma}_p^{e}}\left|\lim_{\substack{z'\to z\\ z'\in-\widetilde{\Sigma}_p^{e}}}\int_{\widetilde{\Sigma}_p^{e}}\frac{(g(s)w_{+}^{\Sigma_{\mathcal{R}}^{e}}(s))_{jl}}{s-z'}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\right|\\
&\leqslant \|g(\cdot)\|_{\mathcal{L}^{\infty}_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}_p^{e})}\max_{j,l=1,2}\sup_{z\in\widetilde{\Sigma}_p^{e}}\left|\lim_{\substack{z'\to z\\ z'\in-\widetilde{\Sigma}_p^{e}}}\int_{\widetilde{\Sigma}_p^{e}}\frac{(w_{+}^{\Sigma_{\mathcal{R}}^{e}}(s))_{jl}}{s-z'}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\right|\\
&\leqslant \|g(\cdot)\|_{\mathcal{L}^{\infty}_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}_p^{e})}\max_{j,l=1,2}\sup_{z\in\widetilde{\Sigma}_p^{e}}\left|\lim_{\substack{z'\to z\\ z'\in-\widetilde{\Sigma}_p^{e}}}\left(\int_{\Sigma_p^{e,1}\setminus\mathbb{U}_0^{e}}+\int_{\mathbb{U}_0^{e}}+\sum_{k=1}^{N+1}\int_{\Sigma_{p,k}^{e,2}}+\int_{\Sigma_p^{e,3}}\right.\right.\\
&+\left.\left.\int_{\Sigma_p^{e,4}}+\sum_{k=1}^{N+1}\left(\int_{\partial\mathbb{U}^{e}_{\delta_{b_{k-1}}}}+\int_{\partial\mathbb{U}^{e}_{a_k}}\right)\right)\frac{(w_{+}^{\Sigma_{\mathcal{R}}^{e}}(s))_{jl}}{s-z'}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\right|\\
&\underset{n\to\infty}{\leqslant} \|g(\cdot)\|_{\mathcal{L}^{\infty}_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}_p^{e})}\max_{j,l=1,2}\sup_{z\in\widetilde{\Sigma}_p^{e}}\left|\lim_{\substack{z'\to z\\ z'\in-\widetilde{\Sigma}_p^{e}}}\left(\int_{\Sigma_p^{e,1}\setminus\mathbb{U}_0^{e}}\frac{(\mathcal{O}(\mathrm{e}^{-nc_{\infty}|s|}f_{\infty}(n)))_{jl}}{s-z'}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\right.\right.\\
&+\int_{\mathbb{U}_0^{e}}\frac{(\mathcal{O}(\mathrm{e}^{-nc_0|s|^{-1}}f_0(n)))_{jl}}{s-z'}\frac{\mathrm{d}s}{2\pi\mathrm{i}}+\sum_{k=1}^{N}\int_{\Sigma_{p,k}^{e,2}}\frac{(\mathcal{O}(\mathrm{e}^{-nc_k(s-a_k^{e})}f_k(n)))_{jl}}{s-z'}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\\
&+\int_{\Sigma_p^{e,3}}\frac{(\mathcal{O}(\mathrm{e}^{-n\widehat{c}|s|}\widehat{f}(n)))_{jl}}{s-z'}\frac{\mathrm{d}s}{2\pi\mathrm{i}}+\int_{\Sigma_p^{e,4}}\frac{(\mathcal{O}(\mathrm{e}^{-n\check{c}|s|}\check{f}(n)))_{jl}}{s-z'}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\\
&+\sum_{k=1}^{N+1}\left(\int_{\partial\mathbb{U}^{e}_{\delta_{b_{k-1}}}}\left(\mathcal{O}\left(\frac{\overset{e}{\mathfrak{M}}{}^{\infty}(s)\left(\begin{smallmatrix}*&*\\ *&*\end{smallmatrix}\right)(\overset{e}{\mathfrak{M}}{}^{\infty}(s))^{-1}}{n(s-z')(s-b^{e}_{k-1})^{3/2}G^{e}_{b_{k-1}}(s)}\right)\right)_{jl}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\right.
\end{aligned}$$

$$+\int_{\partial \mathbb{U}^{e}_{\delta_{a_k}}}\left(\mathcal{O}\left(\frac{\overset{e}{\mathfrak{M}}{}^{\infty}(s)\left(\begin{smallmatrix} * & * \\ * & * \end{smallmatrix}\right)(\overset{e}{\mathfrak{M}}{}^{\infty}(s))^{-1}}{n(s-z')(s-a^{e}_{k})^{3/2}G^{e}_{a_k}(s)}\right)\right)_{jl}\frac{\mathrm{d}s}{2\pi \mathrm{i}}\Bigg)\Bigg|\Bigg|,$$

whence, (cf. Lemma 4.5) using the fact that the respective factors $\gamma^{e}(z)\pm(\gamma^{e}(z))^{-1}$ and $\boldsymbol{\theta}^{e}(\pm \boldsymbol{u}^{e}(z)-\frac{n}{2\pi}\boldsymbol{\Omega}^{e}\pm \boldsymbol{d}_{e})$ are uniformly bounded (with respect to z) in compact intervals outside open intervals surrounding the end-points of the support of the 'even' equilibrium measure, one arrives at, after a straightforward integration argument and an application of the Maximum Length (ML) Theorem,

$$\begin{aligned}||C^{\infty}_{w^{\Sigma^{e}_{\mathcal{R}}}}g||_{\mathcal{L}^{\infty}_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^{e}_{p})} \underset{n\to\infty}{\leqslant} ||g(\cdot)||_{\mathcal{L}^{\infty}_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^{e}_{p})}&\left(\mathcal{O}\left(\frac{f(n)\mathrm{e}^{-nc}}{n\,\mathrm{dist}(z,\widetilde{\Sigma}^{e}_{p})}\right)+\mathcal{O}\left(\frac{f(n)\mathrm{e}^{-nc}}{\mathrm{dist}(z,\widetilde{\Sigma}^{e}_{p})}\right)\right.\\ &\left.+\,\mathcal{O}\left(\frac{f(n)}{n\,\mathrm{dist}(z,\widetilde{\Sigma}^{e}_{p})}\right)\right)\underset{n\to\infty}{=}||g(\cdot)||_{\mathcal{L}^{\infty}_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^{e}_{p})}\mathcal{O}(n^{-1}f(n)),\end{aligned}$$

where $\mathrm{dist}(z,\widetilde{\Sigma}^{e}_{p}):=\inf\left\{|z-r|;\,r\in\widetilde{\Sigma}^{e}_{p},\,z\in\mathbb{C}\setminus\widetilde{\Sigma}^{e}_{p}\right\}$ (>0), and $f(n)=_{n\to\infty}\mathcal{O}(1)$, whence one obtains the asymptotic (as $n\to\infty$) estimate for $||C^{\infty}_{w^{\Sigma^{e}_{\mathcal{R}}}}||_{\mathcal{N}_{\infty}(\widetilde{\Sigma}^{e}_{p})}$ stated in the Proposition. Similarly, for $||C^{\infty}_{w^{\Sigma^{e}_{\mathcal{R}}}}||_{\mathcal{N}_{2}(\widetilde{\Sigma}^{e}_{p})}$:

$$\begin{aligned}||C^{\infty}_{w^{\Sigma^{e}_{\mathcal{R}}}}g||_{\mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^{e}_{p})}:=&\left(\int_{\widetilde{\Sigma}^{e}_{p}}|(C^{\infty}_{w^{\Sigma^{e}_{\mathcal{R}}}}g)(z)|^{2}\,|\mathrm{d}z|\right)^{1/2}=\left(\int_{\widetilde{\Sigma}^{e}_{p}}\sum_{j,l=1}^{2}(C^{\infty}_{w^{\Sigma^{e}_{\mathcal{R}}}}g)_{jl}(z)\overline{(C^{\infty}_{w^{\Sigma^{e}_{\mathcal{R}}}}g)_{jl}(z)}\,|\mathrm{d}z|\right)^{1/2}\\
=&\left(\int_{\widetilde{\Sigma}^{e}_{p}}\sum_{j,l=1}^{2}\left|\lim_{\substack{z'\to z\\ z'\in -\widetilde{\Sigma}^{e}_{p}}}\int_{\widetilde{\Sigma}^{e}_{p}}\frac{(g(s)w^{\Sigma^{e}_{\mathcal{R}}}_{+}(s))_{jl}}{s-z'}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\right|^{2}|\mathrm{d}z|\right)^{1/2}\\
\leqslant&\ ||g(\cdot)||_{\mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^{e}_{p})}\left(\int_{\widetilde{\Sigma}^{e}_{p}}\sum_{j,l=1}^{2}\left|\lim_{\substack{z'\to z\\ z'\in -\widetilde{\Sigma}^{e}_{p}}}\int_{\widetilde{\Sigma}^{e}_{p}}\frac{(w^{\Sigma^{e}_{\mathcal{R}}}_{+}(s))_{jl}}{s-z'}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\right|^{2}|\mathrm{d}z|\right)^{1/2}\\
\leqslant&\ ||g(\cdot)||_{\mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^{e}_{p})}\left(\int_{\widetilde{\Sigma}^{e}_{p}}\sum_{j,l=1}^{2}\left|\lim_{\substack{z'\to z\\ z'\in -\widetilde{\Sigma}^{e}_{p}}}\left(\int_{\Sigma^{e,1}_{p}\setminus\mathbb{U}^{e}_{0}}+\int_{\mathbb{U}^{e}_{0}}+\sum_{k=1}^{N+1}\int_{\Sigma^{e,2}_{p,k}}+\int_{\Sigma^{e,3}_{p}}\right.\right.\right.\\
&\left.\left.\left.+\int_{\Sigma^{e,4}_{p}}+\sum_{k=1}^{N+1}\left(\int_{\partial\mathbb{U}^{e}_{\delta_{b_{k-1}}}}+\int_{\partial\mathbb{U}^{e}_{a_k}}\right)\right)\frac{(w^{\Sigma^{e}_{\mathcal{R}}}_{+}(s))_{jl}}{s-z'}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\right|^{2}|\mathrm{d}z|\right)^{1/2}\\
\underset{n\to\infty}{\leqslant}&\ ||g(\cdot)||_{\mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^{e}_{p})}\left(\int_{\widetilde{\Sigma}^{e}_{p}}\sum_{j,l=1}^{2}\left|\lim_{\substack{z'\to z\\ z'\in -\widetilde{\Sigma}^{e}_{p}}}\left(\int_{\Sigma^{e,1}_{p}\setminus\mathbb{U}^{e}_{0}}\frac{(\mathcal{O}(\mathrm{e}^{-nc_{\infty}|s|}f_{\infty}(n)))_{jl}}{s-z'}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\right.\right.\right.\\
&+\int_{\mathbb{U}^{e}_{0}}\frac{(\mathcal{O}(\mathrm{e}^{-nc_{0}|s|^{-1}}f_{0}(n)))_{jl}}{s-z'}\frac{\mathrm{d}s}{2\pi\mathrm{i}}+\sum_{k=1}^{N}\int_{\Sigma^{e,2}_{p,k}}\frac{(\mathcal{O}(\mathrm{e}^{-nc_{k}(s-a^{e}_{k})}f_{k}(n)))_{jl}}{s-z'}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\\
&+\int_{\Sigma^{e,3}_{p}}\frac{(\mathcal{O}(\mathrm{e}^{-n\widehat{c}|s|}\widehat{f}(n)))_{jl}}{s-z'}\frac{\mathrm{d}s}{2\pi\mathrm{i}}+\int_{\Sigma^{e,4}_{p}}\frac{(\mathcal{O}(\mathrm{e}^{-n\breve{c}|s|}\breve{f}(n)))_{jl}}{s-z'}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\\
&+\sum_{k=1}^{N+1}\left(\int_{\partial\mathbb{U}^{e}_{\delta_{b_{k-1}}}}\left(\mathcal{O}\left(\frac{\overset{e}{\mathfrak{M}}{}^{\infty}(s)\left(\begin{smallmatrix} * & * \\ * & * \end{smallmatrix}\right)(\overset{e}{\mathfrak{M}}{}^{\infty}(s))^{-1}}{n(s-z')(s-b^{e}_{k-1})^{3/2}G^{e}_{b_{k-1}}(s)}\right)\right)_{jl}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\right.\\
&\left.\left.\left.+\int_{\partial\mathbb{U}^{e}_{\delta_{a_k}}}\left(\mathcal{O}\left(\frac{\overset{e}{\mathfrak{M}}{}^{\infty}(s)\left(\begin{smallmatrix} * & * \\ * & * \end{smallmatrix}\right)(\overset{e}{\mathfrak{M}}{}^{\infty}(s))^{-1}}{n(s-z')(s-a^{e}_{k})^{3/2}G^{e}_{a_k}(s)}\right)\right)_{jl}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\right)\right)\right|^{2}|\mathrm{d}z|\right)^{1/2},\end{aligned}$$

whence, (cf. Lemma 4.5) using the fact that the respective factors $\gamma^{e}(z)\pm(\gamma^{e}(z))^{-1}$ and $\boldsymbol{\theta}^{e}(\pm \boldsymbol{u}^{e}(z)-\frac{n}{2\pi}\boldsymbol{\Omega}^{e}\pm \boldsymbol{d}_{e})$ are uniformly bounded (with respect to z) in compact intervals outside open intervals surrounding

the end-points of the support of the 'even' equilibrium measure, one arrives at, after a straightforward integration argument and an application of the ML Theorem,

$$\begin{aligned}\|C^{\infty}_{w^{\Sigma^e_{\mathcal{R}}}} g\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^e_p)} \underset{n\to\infty}{\leqslant} \|g(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^e_p)}\Bigg(\mathcal{O}\left(\frac{f(n)\mathrm{e}^{-nc}}{n\,\mathrm{dist}(z,\widetilde{\Sigma}^e_p)}\right)+\mathcal{O}\left(\frac{f(n)\mathrm{e}^{-nc}}{\mathrm{dist}(z,\widetilde{\Sigma}^e_p)}\right)\\ +\,\mathcal{O}\left(\frac{f(n)}{n\,\mathrm{dist}(z,\widetilde{\Sigma}^e_p)}\right)\Bigg) \underset{n\to\infty}{=} \|g(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^e_p)}\mathcal{O}(n^{-1}f(n)),\end{aligned}$$

where $f(n) =_{n\to\infty} \mathcal{O}(1)$, whence one obtains the asymptotic (as $n \to \infty$) estimate for $\|C^{\infty}_{w^{\Sigma^e_{\mathcal{R}}}}\|_{\mathcal{N}_2(\widetilde{\Sigma}^e_p)}$ stated in the Proposition. The above analysis establishes the fact that, as $n\to\infty$, $C^{\infty}_{w^{\Sigma^e_{\mathcal{R}}}} \in \mathcal{N}_2(\widetilde{\Sigma}^e_p)$, with operator norm $\|C^{\infty}_{w^{\Sigma^e_{\mathcal{R}}}}\|_{\mathcal{N}_2(\widetilde{\Sigma}^e_p)} =_{n\to\infty} \mathcal{O}(n^{-1}f(n))$, where $f(n)=_{n\to\infty}\mathcal{O}(1)$; due to a well-known result for bounded linear operators in Hilbert space [105], it follows, thus, that $(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\mathcal{R}}}})^{-1}\restriction_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^e_p)}$ exists, and $(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\mathcal{R}}}})\restriction_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^e_p)}$ can be inverted by a Neumann series (as $n\to\infty$), with $\|(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\mathcal{R}}}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^e_p)} \leqslant_{n\to\infty} (1-\|C^{\infty}_{w^{\Sigma^e_{\mathcal{R}}}}\|_{\mathcal{N}_2(\widetilde{\Sigma}^e_p)})^{-1} =_{n\to\infty} \mathcal{O}(1)$. □

Lemma 5.2. *Set* $\Sigma^e_{\circlearrowright} := \Sigma^{e,5}_p$ $(=\cup_{j=1}^{N+1}(\partial\mathbb{U}^e_{\delta_{b_{j-1}}} \cup \partial\mathbb{U}^e_{\delta_{a_j}}))$ *and* $\Sigma^e_{\blacksquare} := \widetilde{\Sigma}^e_p \setminus \Sigma^e_{\circlearrowright}$, *and let* $\mathcal{R}^e\colon \mathbb{C}\setminus\widetilde{\Sigma}^e_p \to \mathrm{SL}_2(\mathbb{C})$ *solve the (equivalent)* RHP $(\mathcal{R}^e(z), v^e_{\mathcal{R}}(z), \widetilde{\Sigma}^e_p)$ *formulated in Proposition* 5.2 *with integral representation given by Equation* (5.1). *Let the asymptotic (as $n\to\infty$) estimates and bounds given in Propositions* 5.1 *and* 5.2 *be valid. Then, uniformly for compact subsets of* $\mathbb{C}\setminus\widetilde{\Sigma}^e_p \ni z$,

$$\mathcal{R}^e(z) \underset{\substack{n\to\infty\\ z\in\mathbb{C}\setminus\widetilde{\Sigma}^e_p}}{=} \mathrm{I}+\int_{\Sigma^e_{\circlearrowright}} \frac{w^{\Sigma^e_{\circlearrowright}}_+(s)}{s-z}\,\frac{\mathrm{d}s}{2\pi\mathrm{i}}+\mathcal{O}\left(\frac{f(n)}{n^2\,\mathrm{dist}(z,\widetilde{\Sigma}^e_p)}\right),$$

where $w^{\Sigma^e_{\circlearrowright}}_+(z) := w^{\Sigma^e_{\mathcal{R}}}_+(z)\restriction_{\Sigma^e_{\circlearrowright}}$, *and* $(f(n))_{ij} =_{n\to\infty} \mathcal{O}(1)$, $i,j=1,2$.

Proof. Define $\Sigma^e_{\circlearrowright}$ and $\Sigma^e_{\blacksquare}$ as in the Lemma, and write $\widetilde{\Sigma}^e_p = (\widetilde{\Sigma}^e_p \setminus \Sigma^e_{\circlearrowright}) \cup \Sigma^e_{\circlearrowright} := \Sigma^e_{\blacksquare} \cup \Sigma^e_{\circlearrowright}$ (with $\Sigma^e_{\blacksquare} \cap \Sigma^e_{\circlearrowright} = \varnothing$). Recall, from Equation (5.1), the integral representation for $\mathcal{R}^e\colon \mathbb{C}\setminus\widetilde{\Sigma}^e_p \to \mathrm{SL}_2(\mathbb{C})$:

$$\mathcal{R}^e(z) = \mathrm{I}+\int_{\widetilde{\Sigma}^e_p} \frac{\mu^{\Sigma^e_{\mathcal{R}}}(s)w^{\Sigma^e_{\mathcal{R}}}_+(s)}{s-z}\,\frac{\mathrm{d}s}{2\pi\mathrm{i}}, \quad z\in\mathbb{C}\setminus\widetilde{\Sigma}^e_p.$$

Using the linearity property of the Cauchy integral operator $C^{\infty}_{w^{\Sigma^e_{\mathcal{R}}}}$, one shows that $C^{\infty}_{w^{\Sigma^e_{\mathcal{R}}}} = C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}} + C^{\infty}_{w^{\Sigma^e_{\blacksquare}}}$. Via a repeated application of the second resolvent identity[14]:

$$\begin{aligned}\mu^{\Sigma^e_{\mathcal{R}}}(z) &= \mathrm{I}+((\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\mathcal{R}}}})^{-1}C^{\infty}_{w^{\Sigma^e_{\mathcal{R}}}}\mathrm{I})(z) = \mathrm{I}+((\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}}-C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}(C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}}+C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})\mathrm{I})(z)\\
&= \mathrm{I}+((\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}}-C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}}\mathrm{I})(z)+((\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}}-C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}C^{\infty}_{w^{\Sigma^e_{\blacksquare}}}\mathrm{I})(z)\\
&= \mathrm{I}+(((\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}})(\mathbf{id}-(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}})^{-1}C^{\infty}_{w^{\Sigma^e_{\blacksquare}}}))^{-1}C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}}\mathrm{I})(z)\\
&+ (((\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})(\mathbf{id}-(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}}))^{-1}C^{\infty}_{w^{\Sigma^e_{\blacksquare}}}\mathrm{I})(z)\\
&= \mathrm{I}+((\mathbf{id}-(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}})^{-1}C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}(\mathbf{id}+(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}})^{-1}C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}})C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}}\mathrm{I})(z)\\
&+ ((\mathbf{id}-(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}})^{-1}(\mathbf{id}+(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})C^{\infty}_{w^{\Sigma^e_{\blacksquare}}}\mathrm{I})(z)\\
&= \mathrm{I}+((\mathbf{id}-(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}})^{-1}C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}((\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}})^{-1}C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}})(C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}}\mathrm{I}))(z)\\
&+ ((\mathbf{id}-(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}})^{-1}((\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})(C^{\infty}_{w^{\Sigma^e_{\blacksquare}}}\mathrm{I}))(z)\\
&+ ((\mathbf{id}-(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}})^{-1}C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}(C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}}\mathrm{I}))(z)+((\mathbf{id}-(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}})^{-1}(C^{\infty}_{w^{\Sigma^e_{\blacksquare}}}\mathrm{I}))(z)\\
&= \mathrm{I}+((\mathbf{id}+(\mathbf{id}-(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}})^{-1}C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}})^{-1}C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})(C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}}\mathrm{I}))(z)\end{aligned}$$

[14]For general operators $\mathcal{A}$ and $\mathcal{B}$, if $(\mathbf{id}-\mathcal{A})^{-1}$ and $(\mathbf{id}-\mathcal{B})^{-1}$ exist, then $(\mathbf{id}-\mathcal{B})^{-1}-(\mathbf{id}-\mathcal{A})^{-1} = (\mathbf{id}-\mathcal{B})^{-1}(\mathcal{B}-\mathcal{A})(\mathbf{id}-\mathcal{A})^{-1}$ [105].

$$
\begin{aligned}
&+((\mathbf{id}+(\mathbf{id}-(\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}})^{-1}C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}})^{-1}C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}})(C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}}\mathrm{I}))(z)\\
&+((\mathbf{id}+(\mathbf{id}-(\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}})^{-1}C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}})^{-1}C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}})(\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}})^{-1}C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}}(C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}}\mathrm{I}))(z)\\
&+((\mathbf{id}+(\mathbf{id}-(\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}})^{-1}C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}})^{-1}C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}})(\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}})^{-1}C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}}(C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}}\mathrm{I}))(z)\\
&=\mathrm{I}+(C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}}\mathrm{I})(z)+(C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}}\mathrm{I})(z)+((\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}})^{-1}C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}}(C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}}\mathrm{I}))(z)\\
&+((\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}})^{-1}C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}}(C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}}\mathrm{I}))(z)+((\mathbf{id}-(\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}})^{-1}C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}})^{-1}C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}}\\
&\times(C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}}\mathrm{I}))(z)+((\mathbf{id}-(\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}})^{-1}C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}})^{-1}C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}}(C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}}\mathrm{I}))(z)\\
&+((\mathbf{id}-(\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}})^{-1}C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}})^{-1}C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}}(\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}})^{-1}C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}}(C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}}\mathrm{I}))(z)\\
&+((\mathbf{id}-(\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}})^{-1}C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}})^{-1}C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}}(\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}})^{-1}C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}}(C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}}\mathrm{I}))(z)\\
&=\mathrm{I}+(C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}}\mathrm{I})(z)+(C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}}\mathrm{I})(z)+((\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}})^{-1}C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}}(C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}}\mathrm{I}))(z)+((\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}})^{-1}C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}}\\
&\times(C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}}\mathrm{I}))(z)+((\mathbf{id}-(\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}})^{-1}C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}}C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}})^{-1}\\
&\times C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}}(C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}}\mathrm{I}))(z)+((\mathbf{id}-(\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}})^{-1}C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}}C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}})^{-1}\\
&\times(\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}})^{-1}C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}}(C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}}\mathrm{I}))(z)+((\mathbf{id}-(\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}})^{-1}C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}}C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}})^{-1}\\
&\times(\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}})^{-1}C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}}(\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}})^{-1}C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}}(C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}}\mathrm{I}))(z)+((\mathbf{id}-(\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}})^{-1}\\
&\times(\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}})^{-1}C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}}C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}})^{-1}C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}}(\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}})^{-1}C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}}\\
&\times(C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}}\mathrm{I}))(z);
\end{aligned}
$$

hence, recalling the integral representation for $\mathcal{R}^{e}(z)$ given above, one arrives at, for $\mathbb{C}\setminus\widetilde{\Sigma}^{e}_{p}\ni z$,

$$
\mathcal{R}^{e}(z)-\mathrm{I}-\int_{\Sigma^{e}_{\circlearrowright}}\frac{w^{\Sigma^{e}_{\circlearrowright}}_{+}(s)}{s-z}\,\frac{\mathrm{d}s}{2\pi\mathrm{i}}=\int_{\Sigma^{e}_{\blacksquare}}\frac{w^{\Sigma^{e}_{\blacksquare}}_{+}(s)}{s-z}\,\frac{\mathrm{d}s}{2\pi\mathrm{i}}+\sum_{k=1}^{8}I^{e}_{k}, \tag{5.2}
$$

where $w^{\Sigma^{e}_{\circlearrowright}}_{+}(z):=w^{\Sigma^{e}_{\mathcal{R}}}_{+}(z)\restriction_{\Sigma^{e}_{\circlearrowright}}$, $w^{\Sigma^{e}_{\blacksquare}}_{+}(z):=w^{\Sigma^{e}_{\mathcal{R}}}_{+}(z)\restriction_{\Sigma^{e}_{\blacksquare}}$,

$$
I^{e}_{1}:=\int_{\widetilde{\Sigma}^{e}_{p}}\frac{(C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}}\mathrm{I})(s)w^{\Sigma^{e}_{\mathcal{R}}}_{+}(s)}{s-z}\,\frac{\mathrm{d}s}{2\pi\mathrm{i}},\qquad I^{e}_{2}:=\int_{\widetilde{\Sigma}^{e}_{p}}\frac{(C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}}\mathrm{I})(s)w^{\Sigma^{e}_{\mathcal{R}}}_{+}(s)}{s-z}\,\frac{\mathrm{d}s}{2\pi\mathrm{i}},
$$

$$
I^{e}_{3}:=\int_{\widetilde{\Sigma}^{e}_{p}}\frac{((\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}})^{-1}C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}}(C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}}\mathrm{I}))(s)w^{\Sigma^{e}_{\mathcal{R}}}_{+}(s)}{s-z}\,\frac{\mathrm{d}s}{2\pi\mathrm{i}},
$$

$$
I^{e}_{4}:=\int_{\widetilde{\Sigma}^{e}_{p}}\frac{((\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}})^{-1}C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}}(C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}}\mathrm{I}))(s)w^{\Sigma^{e}_{\mathcal{R}}}_{+}(s)}{s-z}\,\frac{\mathrm{d}s}{2\pi\mathrm{i}},
$$

$$
\begin{aligned}
I^{e}_{5}:=&\int_{\widetilde{\Sigma}^{e}_{p}}((\mathbf{id}-(\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}})^{-1}C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}}C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}})^{-1}\\
&\times\frac{C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}}(C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}}\mathrm{I}))(s)w^{\Sigma^{e}_{\mathcal{R}}}_{+}(s)}{s-z}\,\frac{\mathrm{d}s}{2\pi\mathrm{i}},\\
I^{e}_{6}:=&\int_{\widetilde{\Sigma}^{e}_{p}}((\mathbf{id}-(\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}})^{-1}C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}}C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}})^{-1}\\
&\times\frac{C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}}(C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}}\mathrm{I}))(s)w^{\Sigma^{e}_{\mathcal{R}}}_{+}(s)}{s-z}\,\frac{\mathrm{d}s}{2\pi\mathrm{i}},\\
I^{e}_{7}:=&\int_{\widetilde{\Sigma}^{e}_{p}}((\mathbf{id}-(\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}})^{-1}C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}}C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}})^{-1}\\
&\times\frac{C^{\infty}_{w^{\Sigma^{e}_{\circlearrowright}}}(\mathbf{id}-C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}})^{-1}C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}}(C^{\infty}_{w^{\Sigma^{e}_{\blacksquare}}}\mathrm{I}))(s)w^{\Sigma^{e}_{\mathcal{R}}}_{+}(s)}{s-z}\,\frac{\mathrm{d}s}{2\pi\mathrm{i}},
\end{aligned}
$$

$$I_8^e := \int_{\widetilde{\Sigma}_p^e} ((\mathbf{id}-(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\cup}}})^{-1}C^{\infty}_{w^{\Sigma^e_{\cup}}}C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\cup}}})^{-1}$$
$$\times \frac{C^{\infty}_{w^{\Sigma^e_{\blacksquare}}}(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\cup}}})^{-1}C^{\infty}_{w^{\Sigma^e_{\cup}}}(C^{\infty}_{w^{\Sigma^e_{\cup}}}\mathrm{I}))(s)w_+^{\Sigma^e_{\mathcal{R}}}(s)}{s-z}\frac{\mathrm{d}s}{2\pi\mathrm{i}}.$$

One now proceeds to estimate, as $n\to\infty$, the respective terms on the right-hand side of Equation (5.2) using the estimates and bounds given in Propositions 5.1 and 5.2.

$$\left|\int_{\Sigma^e_{\blacksquare}} \frac{w_+^{\Sigma^e_{\blacksquare}}(s)}{s-z}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\right| \leqslant \int_{\Sigma^e_{\blacksquare}} \frac{|w_+^{\Sigma^e_{\blacksquare}}(s)|}{|s-z|}\frac{|\mathrm{d}s|}{2\pi} \leqslant \frac{\|w_+^{\Sigma^e_{\blacksquare}}(\cdot)\|_{\mathcal{L}^1_{\mathrm{M}_2(\mathbb{C})}(\Sigma^e_{\blacksquare})}}{2\pi\,\mathrm{dist}(z,\Sigma^e_{\blacksquare})} \underset{n\to\infty}{\leqslant} \mathcal{O}\left(\frac{f(n)\mathrm{e}^{-nc}}{n\,\mathrm{dist}(z,\widetilde{\Sigma}^e_p)}\right),$$

where, here and below, ($f(n)>0$ and) $f(n)=_{n\to\infty}\mathcal{O}(1)$ and $c>0$. One estimates I_1^e as follows:

$$\begin{aligned}
|I_1^e| &\leqslant \int_{\widetilde{\Sigma}^e_p} \frac{|(C^{\infty}_{w^{\Sigma^e_{\blacksquare}}}\mathrm{I})(s)||w_+^{\Sigma^e_{\mathcal{R}}}(s)|}{|s-z|}\frac{|\mathrm{d}s|}{2\pi} \leqslant \frac{\|(C^{\infty}_{w^{\Sigma^e_{\blacksquare}}}\mathrm{I})(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^e_p)}\|w_+^{\Sigma^e_{\mathcal{R}}}(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^e_p)}}{2\pi\,\mathrm{dist}(z,\widetilde{\Sigma}^e_p)}\\
&\leqslant \frac{\mathrm{const.}\,\|w_+^{\Sigma^e_{\blacksquare}}(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Sigma^e_{\blacksquare})}(\|w_+^{\Sigma^e_{\cup}}(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Sigma^e_{\cup})}+\|w_+^{\Sigma^e_{\blacksquare}}(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Sigma^e_{\blacksquare})})}{2\pi\,\mathrm{dist}(z,\widetilde{\Sigma}^e_p)}\\
&\underset{n\to\infty}{\leqslant} \mathcal{O}\left(\frac{f(n)\mathrm{e}^{-nc}}{\sqrt{n}\,\mathrm{dist}(z,\widetilde{\Sigma}^e_p)}\right)\left(\mathcal{O}\left(\frac{f(n)}{n}\right)+\mathcal{O}\left(\frac{f(n)\mathrm{e}^{-nc}}{\sqrt{n}}\right)\right) \underset{n\to\infty}{\leqslant} \mathcal{O}\left(\frac{f(n)\mathrm{e}^{-nc}}{n\,\mathrm{dist}(z,\widetilde{\Sigma}^e_p)}\right),
\end{aligned}$$

where, here and below, const. denotes some positive, $\mathcal{O}(1)$ constant; in going from the second line to the third line in the above asymptotic (as $n\to\infty$) estimation for I_1^e, one uses the fact that, for $a,b>0$, $\sqrt{a^2+b^2}\leqslant\sqrt{a^2}+\sqrt{b^2}$ (a fact used repeatedly below). One estimates I_2^e as follows:

$$\begin{aligned}
|I_2^e| &\leqslant \int_{\widetilde{\Sigma}^e_p} \frac{|(C^{\infty}_{w^{\Sigma^e_{\cup}}}\mathrm{I})(s)||w_+^{\Sigma^e_{\mathcal{R}}}(s)|}{|s-z|}\frac{|\mathrm{d}s|}{2\pi} \leqslant \frac{\|(C^{\infty}_{w^{\Sigma^e_{\cup}}}\mathrm{I})(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^e_p)}\|w_+^{\Sigma^e_{\mathcal{R}}}(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^e_p)}}{2\pi\,\mathrm{dist}(z,\widetilde{\Sigma}^e_p)}\\
&\leqslant \frac{\mathrm{const.}\,\|w_+^{\Sigma^e_{\cup}}(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Sigma^e_{\cup})}(\|w_+^{\Sigma^e_{\cup}}(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Sigma^e_{\cup})}+\|w_+^{\Sigma^e_{\blacksquare}}(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Sigma^e_{\blacksquare})})}{2\pi\,\mathrm{dist}(z,\widetilde{\Sigma}^e_p)}\\
&\underset{n\to\infty}{\leqslant} \mathcal{O}\left(\frac{f(n)}{n\,\mathrm{dist}(z,\widetilde{\Sigma}^e_p)}\right)\left(\mathcal{O}\left(\frac{f(n)}{n}\right)+\mathcal{O}\left(\frac{f(n)\mathrm{e}^{-nc}}{\sqrt{n}}\right)\right) \underset{n\to\infty}{\leqslant} \mathcal{O}\left(\frac{f(n)}{n^2\,\mathrm{dist}(z,\widetilde{\Sigma}^e_p)}\right).
\end{aligned}$$

One estimates I_3^e as follows:

$$\begin{aligned}
|I_3^e| &\leqslant \int_{\widetilde{\Sigma}^e_p} \frac{|((\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}C^{\infty}_{w^{\Sigma^e_{\blacksquare}}}(C^{\infty}_{w^{\Sigma^e_{\blacksquare}}}\mathrm{I}))(s)||w_+^{\Sigma^e_{\mathcal{R}}}(s)|}{|s-z|}\frac{|\mathrm{d}s|}{2\pi}\\
&\leqslant \frac{\|((\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}C^{\infty}_{w^{\Sigma^e_{\blacksquare}}}(C^{\infty}_{w^{\Sigma^e_{\blacksquare}}}\mathrm{I}))(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^e_p)}\|w_+^{\Sigma^e_{\mathcal{R}}}(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^e_p)}}{2\pi\,\mathrm{dist}(z,\widetilde{\Sigma}^e_p)}\\
&\leqslant \frac{\|(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^e_p)}\|C^{\infty}_{w^{\Sigma^e_{\blacksquare}}}\|_{\mathcal{N}_2(\widetilde{\Sigma}^e_p)}\|(C^{\infty}_{w^{\Sigma^e_{\blacksquare}}}\mathrm{I})(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^e_p)}\|w_+^{\Sigma^e_{\mathcal{R}}}(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^e_p)}}{2\pi\,\mathrm{dist}(z,\widetilde{\Sigma}^e_p)}\\
&\leqslant \frac{\mathrm{const.}\,\|(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^e_p)}\|C^{\infty}_{w^{\Sigma^e_{\blacksquare}}}\|_{\mathcal{N}_2(\widetilde{\Sigma}^e_p)}\|w_+^{\Sigma^e_{\blacksquare}}(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Sigma^e_{\blacksquare})}}{2\pi\,\mathrm{dist}(z,\widetilde{\Sigma}^e_p)}\\
&\times\left(\|w_+^{\Sigma^e_{\cup}}(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Sigma^e_{\cup})}+\|w_+^{\Sigma^e_{\blacksquare}}(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Sigma^e_{\blacksquare})}\right);
\end{aligned}$$

using the fact that (cf. Proposition 5.2) $\|(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^e_p)} =_{n\to\infty} \mathcal{O}(1)$ (via a Neuman series inversion argument, since $\|C^{\infty}_{w^{\Sigma^e_{\blacksquare}}}\|_{\mathcal{N}_2(\widetilde{\Sigma}^e_p)} =_{n\to\infty} \mathcal{O}(n^{-1}f(n)\mathrm{e}^{-nc})$), one gets that

$$|I^e_3| \underset{n\to\infty}{\leqslant} \mathcal{O}\left(\frac{f(n)\mathrm{e}^{-nc}}{n\,\mathrm{dist}(z,\widetilde{\Sigma}^e_p)}\right)\mathcal{O}\left(\frac{f(n)\mathrm{e}^{-nc}}{\sqrt{n}}\right)\left(\mathcal{O}\left(\frac{f(n)}{n}\right)+\mathcal{O}\left(\frac{f(n)\mathrm{e}^{-nc}}{\sqrt{n}}\right)\right) \underset{n\to\infty}{\leqslant} \mathcal{O}\left(\frac{f(n)\mathrm{e}^{-nc}}{n^2\,\mathrm{dist}(z,\widetilde{\Sigma}^e_p)}\right).$$

One estimates I^e_4 as follows:

$$\begin{aligned}
|I^e_4| &\leqslant \int_{\widetilde{\Sigma}^e_p} \frac{|((\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\circlearrowleft}}})^{-1}C^{\infty}_{w^{\Sigma^e_{\circlearrowleft}}}(C^{\infty}_{w^{\Sigma^e_{\circlearrowleft}}}\mathrm{I}))(s)||w_+^{\Sigma^e_{\mathcal{R}}}(s)|}{|s-z|}\,\frac{|\mathrm{d}s|}{2\pi}\\
&\leqslant \frac{\|((\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\circlearrowleft}}})^{-1}C^{\infty}_{w^{\Sigma^e_{\circlearrowleft}}}(C^{\infty}_{w^{\Sigma^e_{\circlearrowleft}}}\mathrm{I}))(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^e_p)}\|w_+^{\Sigma^e_{\mathcal{R}}}(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^e_p)}}{2\pi\,\mathrm{dist}(z,\widetilde{\Sigma}^e_p)}\\
&\leqslant \frac{\|(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\circlearrowleft}}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^e_p)}\|C^{\infty}_{w^{\Sigma^e_{\circlearrowleft}}}\|_{\mathcal{N}_2(\widetilde{\Sigma}^e_p)}\|(C^{\infty}_{w^{\Sigma^e_{\circlearrowleft}}}\mathrm{I})(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^e_p)}\|w_+^{\Sigma^e_{\mathcal{R}}}(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^e_p)}}{2\pi\,\mathrm{dist}(z,\widetilde{\Sigma}^e_p)}\\
&\leqslant \frac{\mathrm{const.}\,\|(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\circlearrowleft}}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^e_p)}\|C^{\infty}_{w^{\Sigma^e_{\circlearrowleft}}}\|_{\mathcal{N}_2(\widetilde{\Sigma}^e_p)}\|w_+^{\Sigma^e_{\circlearrowleft}}(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Sigma^e_{\circlearrowleft})}}{2\pi\,\mathrm{dist}(z,\widetilde{\Sigma}^e_p)}\\
&\quad\times\left(\|w_+^{\Sigma^e_{\circlearrowleft}}(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Sigma^e_{\circlearrowleft})}+\|w_+^{\Sigma^e_{\blacksquare}}(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Sigma^e_{\blacksquare})}\right);
\end{aligned}$$

using the fact that (cf. Proposition 5.2) $\|(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\circlearrowleft}}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^e_p)} =_{n\to\infty} \mathcal{O}(1)$ (via a Neuman series inversion argument, since $\|C^{\infty}_{w^{\Sigma^e_{\circlearrowleft}}}\|_{\mathcal{N}_2(\widetilde{\Sigma}^e_p)} =_{n\to\infty} \mathcal{O}(n^{-1}f(n))$), one gets that

$$|I^e_4| \underset{n\to\infty}{\leqslant} \mathcal{O}\left(\frac{f(n)}{n\,\mathrm{dist}(z,\widetilde{\Sigma}^e_p)}\right)\mathcal{O}\left(\frac{f(n)}{n}\right)\left(\mathcal{O}\left(\frac{f(n)}{n}\right)+\mathcal{O}\left(\frac{f(n)\mathrm{e}^{-nc}}{\sqrt{n}}\right)\right) \underset{n\to\infty}{\leqslant} \mathcal{O}\left(\frac{f(n)}{n^3\,\mathrm{dist}(z,\widetilde{\Sigma}^e_p)}\right).$$

One estimates I^e_5 as follows:

$$\begin{aligned}
|I^e_5| \leqslant& \int_{\widetilde{\Sigma}^e_p} |((\mathbf{id}-(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\circlearrowleft}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}C^{\infty}_{w^{\Sigma^e_{\blacksquare}}}C^{\infty}_{w^{\Sigma^e_{\circlearrowleft}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\circlearrowleft}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}\\
&\times \frac{C^{\infty}_{w^{\Sigma^e_{\circlearrowleft}}}(C^{\infty}_{w^{\Sigma^e_{\blacksquare}}}\mathrm{I}))(s)||w_+^{\Sigma^e_{\mathcal{R}}}(s)|}{|s-z|}\,\frac{|\mathrm{d}s|}{2\pi}\\
\leqslant& \|((\mathbf{id}-(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\circlearrowleft}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}C^{\infty}_{w^{\Sigma^e_{\blacksquare}}}C^{\infty}_{w^{\Sigma^e_{\circlearrowleft}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\circlearrowleft}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}\\
&\times \frac{C^{\infty}_{w^{\Sigma^e_{\circlearrowleft}}}(C^{\infty}_{w^{\Sigma^e_{\blacksquare}}}\mathrm{I}))(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^e_p)}\|w_+^{\Sigma^e_{\mathcal{R}}}(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^e_p)}}{2\pi\,\mathrm{dist}(z,\widetilde{\Sigma}^e_p)}\\
\leqslant& \|(\mathbf{id}-(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\circlearrowleft}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}C^{\infty}_{w^{\Sigma^e_{\blacksquare}}}C^{\infty}_{w^{\Sigma^e_{\circlearrowleft}}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^e_p)}\|(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\circlearrowleft}}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^e_p)}\\
&\times \frac{\|(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^e_p)}\|C^{\infty}_{w^{\Sigma^e_{\circlearrowleft}}}\|_{\mathcal{N}_2(\widetilde{\Sigma}^e_p)}\|(C^{\infty}_{w^{\Sigma^e_{\blacksquare}}}\mathrm{I})(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^e_p)}\|w_+^{\Sigma^e_{\mathcal{R}}}(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^e_p)}}{2\pi\,\mathrm{dist}(z,\widetilde{\Sigma}^e_p)}\\
\leqslant& \|(\mathbf{id}-(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\circlearrowleft}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}C^{\infty}_{w^{\Sigma^e_{\blacksquare}}}C^{\infty}_{w^{\Sigma^e_{\circlearrowleft}}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^e_p)}\|(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\circlearrowleft}}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^e_p)}\\
&\times \mathrm{const.}\,\|(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^e_p)}\|C^{\infty}_{w^{\Sigma^e_{\circlearrowleft}}}\|_{\mathcal{N}_2(\widetilde{\Sigma}^e_p)}\|w_+^{\Sigma^e_{\blacksquare}}(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Sigma^e_{\blacksquare})}\\
&\times \frac{(\|w_+^{\Sigma^e_{\circlearrowleft}}(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Sigma^e_{\circlearrowleft})}+\|w_+^{\Sigma^e_{\blacksquare}}(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Sigma^e_{\blacksquare})})}{2\pi\,\mathrm{dist}(z,\widetilde{\Sigma}^e_p)};
\end{aligned}$$

using the fact that (cf. Proposition 5.2) $\|(\mathbf{id}-(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}C^{\infty}_{w^{\Sigma^e_{\blacksquare}}}C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^e_p)} =_{n\to\infty} \mathcal{O}(1)$ (via a Neuman series inversion argument, since $\|C^{\infty}_{w^{\Sigma^e_{\blacksquare}}}\|_{\mathcal{N}_2(\widetilde{\Sigma}^e_p)} =_{n\to\infty} \mathcal{O}(n^{-1}f(n)\mathrm{e}^{-nc})$ and $\|C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}}\|_{\mathcal{N}_2(\widetilde{\Sigma}^e_p)} =_{n\to\infty} \mathcal{O}(n^{-1}f(n))$), one gets that

$$|I^e_5| \underset{n\to\infty}{\leqslant} \mathcal{O}\left(\frac{f(n)}{n\,\mathrm{dist}(z,\widetilde{\Sigma}^e_p)}\right)\mathcal{O}\left(\frac{f(n)\mathrm{e}^{-nc}}{\sqrt{n}}\right)\left(\mathcal{O}\left(\frac{f(n)}{n}\right)+\mathcal{O}\left(\frac{f(n)\mathrm{e}^{-nc}}{\sqrt{n}}\right)\right) \underset{n\to\infty}{\leqslant} \mathcal{O}\left(\frac{f(n)\mathrm{e}^{-nc}}{n^2\,\mathrm{dist}(z,\widetilde{\Sigma}^e_p)}\right).$$

One estimates I^e_6 as follows:

$$\begin{aligned}
|I^e_6| \leqslant{}& \int_{\widetilde{\Sigma}^e_p} |((\mathbf{id}-(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}})^{-1}C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}}C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}})^{-1}\\
&\times \frac{C^{\infty}_{w^{\Sigma^e_{\blacksquare}}}(C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}}\mathrm{I}))(s)||w^{\Sigma^e_{\mathcal{R}}}_+(s)|}{|s-z|}\,\frac{|\mathrm{d}s|}{2\pi}\\
\leqslant{}& \|((\mathbf{id}-(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}})^{-1}C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}}C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}})^{-1}\\
&\times \frac{C^{\infty}_{w^{\Sigma^e_{\blacksquare}}}(C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}}\mathrm{I}))(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^e_p)}\|w^{\Sigma^e_{\mathcal{R}}}_+(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^e_p)}}{2\pi\,\mathrm{dist}(z,\widetilde{\Sigma}^e_p)}\\
\leqslant{}& \|(\mathbf{id}-(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}})^{-1}C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}}C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^e_p)}\|(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^e_p)}\\
&\times \frac{\|(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^e_p)}\|C^{\infty}_{w^{\Sigma^e_{\blacksquare}}}\|_{\mathcal{N}_2(\widetilde{\Sigma}^e_p)}\|(C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}}\mathrm{I})(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^e_p)}\|w^{\Sigma^e_{\mathcal{R}}}_+(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^e_p)}}{2\pi\,\mathrm{dist}(z,\widetilde{\Sigma}^e_p)}\\
\leqslant{}& \|(\mathbf{id}-(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}})^{-1}C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}}C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^e_p)}\|(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^e_p)}\\
&\times \text{const.}\,\|(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^e_p)}\|C^{\infty}_{w^{\Sigma^e_{\blacksquare}}}\|_{\mathcal{N}_2(\widetilde{\Sigma}^e_p)}\|w^{\Sigma^e_{\circlearrowright}}_+(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Sigma^e_{\circlearrowright})}\\
&\times \frac{(\|w^{\Sigma^e_{\circlearrowright}}_+(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Sigma^e_{\circlearrowright})}+\|w^{\Sigma^e_{\blacksquare}}_+(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Sigma^e_{\blacksquare})})}{2\pi\,\mathrm{dist}(z,\widetilde{\Sigma}^e_p)};
\end{aligned}$$

using the fact that (cf. Proposition 5.2) $\|(\mathbf{id}-(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}})^{-1}C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}}C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^e_p)} =_{n\to\infty} \mathcal{O}(1)$ (via a Neuman series inversion argument, since $\|C^{\infty}_{w^{\Sigma^e_{\blacksquare}}}\|_{\mathcal{N}_2(\widetilde{\Sigma}^e_p)} =_{n\to\infty} \mathcal{O}(n^{-1}f(n)\mathrm{e}^{-nc})$ and $\|C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}}\|_{\mathcal{N}_2(\widetilde{\Sigma}^e_p)} =_{n\to\infty} \mathcal{O}(n^{-1}f(n))$), one gets that

$$|I^e_6| \underset{n\to\infty}{\leqslant} \mathcal{O}\left(\frac{f(n)\mathrm{e}^{-nc}}{n\,\mathrm{dist}(z,\widetilde{\Sigma}^e_p)}\right)\mathcal{O}\left(\frac{f(n)}{n}\right)\left(\mathcal{O}\left(\frac{f(n)}{n}\right)+\mathcal{O}\left(\frac{f(n)\mathrm{e}^{-nc}}{\sqrt{n}}\right)\right) \underset{n\to\infty}{\leqslant} \mathcal{O}\left(\frac{f(n)\mathrm{e}^{-nc}}{n^3\,\mathrm{dist}(z,\widetilde{\Sigma}^e_p)}\right).$$

One estimates I^e_7, succinctly, as follows:

$$\begin{aligned}
|I^e_7| \leqslant{}& \int_{\widetilde{\Sigma}^e_p} |((\mathbf{id}-(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}C^{\infty}_{w^{\Sigma^e_{\blacksquare}}}C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}\\
&\times \frac{C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}}(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}C^{\infty}_{w^{\Sigma^e_{\blacksquare}}}(C^{\infty}_{w^{\Sigma^e_{\blacksquare}}}\mathrm{I}))(s)||w^{\Sigma^e_{\mathcal{R}}}_+(s)|}{|s-z|}\,\frac{|\mathrm{d}s|}{2\pi}\\
\leqslant{}& \frac{\|(\mathbf{id}-(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}C^{\infty}_{w^{\Sigma^e_{\blacksquare}}}C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^e_p)}\|(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^e_p)}}{2\pi\,\mathrm{dist}(z,\widetilde{\Sigma}^e_p)}\\
&\times \|(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^e_p)}\|C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}}\|_{\mathcal{N}_2(\widetilde{\Sigma}^e_p)}\|(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^e_p)}\|C^{\infty}_{w^{\Sigma^e_{\blacksquare}}}\|_{\mathcal{N}_2(\widetilde{\Sigma}^e_p)}\\
&\times \text{const.}\,\|w^{\Sigma^e_{\blacksquare}}_+(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Sigma^e_{\blacksquare})}\left(\|w^{\Sigma^e_{\blacksquare}}_+(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Sigma^e_{\blacksquare})}+\|w^{\Sigma^e_{\circlearrowright}}_+(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Sigma^e_{\circlearrowright})}\right);
\end{aligned}$$

using the fact that (established above) $\|(\mathbf{id}-(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}C^{\infty}_{w^{\Sigma^e_{\blacksquare}}}C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^e_p)} =_{n\to\infty} \mathcal{O}(1)$, one

gets that

$$|I_7^e| \underset{n\to\infty}{\leqslant} \mathcal{O}\left(\frac{f(n)}{n\,\mathrm{dist}(z,\widetilde{\Sigma}_p^e)}\right)\mathcal{O}\left(\frac{f(n)\mathrm{e}^{-nc}}{n}\right)\mathcal{O}\left(\frac{f(n)\mathrm{e}^{-nc}}{\sqrt{n}}\right)\left(\mathcal{O}\left(\frac{f(n)}{n}\right)+\mathcal{O}\left(\frac{f(n)\mathrm{e}^{-nc}}{\sqrt{n}}\right)\right)$$
$$\underset{n\to\infty}{\leqslant} \mathcal{O}\left(\frac{f(n)\mathrm{e}^{-nc}}{n^3\,\mathrm{dist}(z,\widetilde{\Sigma}_p^e)}\right).$$

One estimates I_8^e, succinctly, as follows:

$$|I_8^e| \leqslant \int_{\widetilde{\Sigma}_p^e} |((\mathbf{id}-(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}})^{-1}C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}}C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}})^{-1}$$
$$\times \frac{C^{\infty}_{w^{\Sigma^e_{\blacksquare}}}(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}})^{-1}C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}}(C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}}\mathrm{I}))(s)||w_+^{\Sigma^e_{\mathcal{R}}}(s)|}{|s-z|}\,\frac{|\mathrm{d}s|}{2\pi}$$
$$\leqslant \frac{||(\mathbf{id}-(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}})^{-1}C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}}C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}||_{\mathcal{N}_2(\widetilde{\Sigma}_p^e)}||(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}||_{\mathcal{N}_2(\widetilde{\Sigma}_p^e)}}{2\pi\,\mathrm{dist}(z,\widetilde{\Sigma}_p^e)}$$
$$\times ||(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}})^{-1}||_{\mathcal{N}_2(\widetilde{\Sigma}_p^e)}||C^{\infty}_{w^{\Sigma^e_{\blacksquare}}}||_{\mathcal{N}_2(\widetilde{\Sigma}_p^e)}||(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}})^{-1}||_{\mathcal{N}_2(\widetilde{\Sigma}_p^e)}||C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}}||_{\mathcal{N}_2(\widetilde{\Sigma}_p^e)}$$
$$\times \text{ const.}\,||w_+^{\Sigma^e_{\circlearrowright}}(\cdot)||_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Sigma^e_{\circlearrowright})}\left(||w_+^{\Sigma^e_{\blacksquare}}(\cdot)||_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Sigma^e_{\blacksquare})}+||w_+^{\Sigma^e_{\circlearrowright}}(\cdot)||_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Sigma^e_{\circlearrowright})}\right);$$

using the fact that (established above) $||(\mathbf{id}-(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}(\mathbf{id}-C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}})^{-1}C^{\infty}_{w^{\Sigma^e_{\circlearrowright}}}C^{\infty}_{w^{\Sigma^e_{\blacksquare}}})^{-1}||_{\mathcal{N}_2(\widetilde{\Sigma}_p^e)} =_{n\to\infty} \mathcal{O}(1)$, one gets that

$$|I_8^e| \underset{n\to\infty}{\leqslant} \mathcal{O}\left(\frac{f(n)\mathrm{e}^{-nc}}{n\,\mathrm{dist}(z,\widetilde{\Sigma}_p^e)}\right)\mathcal{O}\left(\frac{f(n)}{n}\right)\mathcal{O}\left(\frac{f(n)}{n}\right)\left(\mathcal{O}\left(\frac{f(n)}{n}\right)+\mathcal{O}\left(\frac{f(n)\mathrm{e}^{-nc}}{\sqrt{n}}\right)\right)$$
$$\underset{n\to\infty}{\leqslant} \mathcal{O}\left(\frac{f(n)\mathrm{e}^{-nc}}{n^4\,\mathrm{dist}(z,\widetilde{\Sigma}_p^e)}\right).$$

Gathering the above-derived bounds, one arrives at the result stated in the Lemma. □

Lemma 5.3. *Let $\mathcal{R}^e\colon \mathbb{C}\setminus\widetilde{\Sigma}_p^e \to \mathrm{SL}_2(\mathbb{C})$ be the solution of the* RHP $(\mathcal{R}^e(z), v^e_{\mathcal{R}}(z), \widetilde{\Sigma}_p^e)$ *formulated in Proposition* 5.2 *with the $n\to\infty$ integral representation given in Lemma* 5.2. *Then, uniformly for compact subsets of $\mathbb{C}\setminus\widetilde{\Sigma}_p^e \ni z$,*

$$\mathcal{R}^e(z) \underset{\substack{n\to\infty\\ z\in\mathbb{C}\setminus\widetilde{\Sigma}_p^e}}{=} \mathrm{I}+\frac{1}{n}\left(\mathcal{R}^e_{\infty}(z)-\widetilde{\mathcal{R}}^e_{\infty}(z)\right)+\mathcal{O}\left(\frac{f(z;n)}{n^2}\right),$$

where $\mathcal{R}^e_{\infty}(z)$ is defined in Theorem 2.3.1, *Equations* (2.23)–(2.57), *$\widetilde{\mathcal{R}}^e_{\infty}(z)$ is defined in Theorem* 2.3.1, *Equations* (2.14)–(2.20) *and* (2.70)–(2.74), *and $f(z;n)$, where the n-dependence arises due to the n-dependence of the associated Riemann theta functions, is a bounded (with respect to z and n),* $\mathrm{GL}_2(\mathbb{C})$*-valued function which is analytic (with respect to z) for $z\in\mathbb{C}\setminus\widetilde{\Sigma}_p^e$, and $(f(\cdot;n))_{kl} =_{n\to\infty} \mathcal{O}(1)$, $k,l=1,2$.*

Remark 5.1. Note from the formulation of Lemma 5.3 above that (cf. Theorem 2.3.1, Equations (2.24)–(2.27)), for $j=1,\dots,N+1$, $\mathrm{tr}(\mathcal{A}^e(a_j^e))=\mathrm{tr}(\mathcal{A}^e(b_{j-1}^e))=\mathrm{tr}(\mathcal{B}^e(a_j^e))=\mathrm{tr}(\mathcal{B}^e(b_{j-1}^e))=0$. ■

Proof. Recall the integral representation for $\mathcal{R}^e\colon \mathbb{C}\setminus\widetilde{\Sigma}_p^e\to\mathrm{SL}_2(\mathbb{C})$ given in Lemma 5.2:

$$\mathcal{R}^e(z) \underset{n\to\infty}{=} \mathrm{I}+\int_{\Sigma^e_{\circlearrowright}}\frac{w_+^{\Sigma^e_{\circlearrowright}}(s)}{s-z}\,\frac{\mathrm{d}s}{2\pi\mathrm{i}}+\mathcal{O}\left(\frac{f(n)}{n^2\,\mathrm{dist}(z,\widetilde{\Sigma}_p^e)}\right), \quad z\in\mathbb{C}\setminus\widetilde{\Sigma}_p^e,$$

where $\Sigma^e_{\circlearrowright} := \cup_{j=1}^{N+1}(\partial\mathbb{U}^e_{\delta_{b_{j-1}}}\cup\partial\mathbb{U}^e_{\delta_{a_j}})$, and $(f(n))_{kl} =_{n\to\infty} \mathcal{O}(1)$, $k,l=1,2$. Recalling that the radii of the open discs $\mathbb{U}^e_{\delta_{b_{j-1}}}, \mathbb{U}^e_{\delta_{a_j}}$, $j=1,\dots,N+1$, are chosen, amongst other factors (cf. Lemmas 4.6 and 4.7), so that

$\mathbb{U}^e_{\delta_{b_{j-1}}} \cap \mathbb{U}^e_{\delta_{a_k}} = \varnothing$, $j,k=1,\dots,N+1$, it follows from the above integral representation that

$$\mathcal{R}^e(z) \underset{n\to\infty}{=} \mathrm{I} - \sum_{j=1}^{N+1}\left(\oint_{\partial\mathbb{U}^e_{\delta_{b_{j-1}}}} + \oint_{\partial\mathbb{U}^e_{\delta_{a_j}}}\right)\frac{w_+^{\Sigma^e_{\circlearrowright}}(s)}{s-z}\,\frac{\mathrm{d}s}{2\pi\mathrm{i}} + \mathcal{O}\left(\frac{f(n)}{n^2\,\mathrm{dist}(z,\widetilde{\Sigma}^e_p)}\right), \quad z\in\mathbb{C}\setminus\widetilde{\Sigma}^e_p,$$

where $\oint_{\partial\mathbb{U}^e_{\delta_{b_{j-1}}}}, \oint_{\partial\mathbb{U}^e_{\delta_{a_j}}}$, $j=1,\dots,N+1$, are counter-clockwise-oriented, closed (contour) integrals (Figure 10) about the end-points of the support of the 'even' equilibrium measure, $\{b^e_{j-1},a^e_j\}_{j=1}^{N+1}$. The evaluation of these $2(N+1)$ contour integrals requires the application of the Cauchy and Residue Theorems; and, since the evaluation of the respective integrals entails analogous calculations, consider, say, and without loss of generality, the evaluation of the integrals about the right-most end-points a^e_j, $j=1,\dots,N$, namely:

$$\oint_{\partial\mathbb{U}^e_{\delta_{a_j}}}\frac{w_+^{\Sigma^e_{\circlearrowright}}(s)}{s-z}\,\frac{\mathrm{d}s}{2\pi\mathrm{i}}, \quad j=1,\dots,N.$$

Recalling from Lemma 4.7 that $\xi^e_{a_j}(z)=(z-a^e_j)^{3/2}G^e_{a_j}(z)$, $z\in\mathbb{U}^e_{\delta_{a_j}}\setminus(-\infty,a^e_j)$, $j=1,\dots,N$, it follows from item (5) of Proposition 5.1 that, since $w_+^{\Sigma^e_{\circlearrowright}}(z)=v^e_{\mathcal{R}}(z)-\mathrm{I}$,

$$\begin{aligned} w_+^{\Sigma^e_{\circlearrowright}}(z) \underset{\substack{n\to\infty\\ z\in\mathbb{C}_\pm\cap\partial\mathbb{U}^e_{\delta_{a_j}}}}{=} & \frac{1}{n(z-a^e_j)^{3/2}G^e_{a_j}(z)}\overset{e}{\mathfrak{M}}{}^{\infty}(z)\begin{pmatrix} \mp(s_1+t_1) & \pm\mathrm{i}(s_1-t_1)\mathrm{e}^{\mathrm{i}n\Omega^e_j} \\ \pm\mathrm{i}(s_1-t_1)\mathrm{e}^{-\mathrm{i}n\Omega^e_j} & \pm(s_1+t_1)\end{pmatrix}(\overset{e}{\mathfrak{M}}{}^{\infty}(z))^{-1} \\ & + \mathcal{O}\left(\frac{1}{n^2(z-a^e_j)^3(G^e_{a_j}(z))^2}\overset{e}{\mathfrak{M}}{}^{\infty}(z)f^e_{a_j}(n)(\overset{e}{\mathfrak{M}}{}^{\infty}(z))^{-1}\right), \quad j=1,\dots,N, \end{aligned}$$

where $\overset{e}{\mathfrak{M}}{}^{\infty}(z)$ and Ω^e_j are defined in Lemma 4.5, and $(f^e_{a_j}(n))_{kl}=_{n\to\infty}\mathcal{O}(1)$, $k,l=1,2$. A matrix-multiplication argument shows that $\overset{e}{\mathfrak{M}}{}^{\infty}(z)\left(\begin{smallmatrix} \mp(s_1+t_1) & \pm\mathrm{i}(s_1-t_1)\mathrm{e}^{\mathrm{i}n\Omega^e_j} \\ \pm\mathrm{i}(s_1-t_1)\mathrm{e}^{-\mathrm{i}n\Omega^e_j} & \pm(s_1+t_1)\end{smallmatrix}\right)(\overset{e}{\mathfrak{M}}{}^{\infty}(z))^{-1}$ is given by

$$\begin{pmatrix} \boxed{\begin{matrix} \mp\frac{1}{4}(s_1+t_1)\left(\frac{(\gamma^e(z))^2+1}{\gamma^e(z)}\right)^2\mathfrak{m}^e_{11}(z)\mathfrak{m}^e_{22}(z) \\ \mp\frac{1}{4}(s_1+t_1)\left(\frac{(\gamma^e(z))^2-1}{\gamma^e(z)}\right)^2\mathfrak{m}^e_{12}(z)\mathfrak{m}^e_{21}(z) \\ \mp\frac{1}{4}(s_1-t_1)\left(\frac{(\gamma^e(z))^4-1}{(\gamma^e(z))^2}\right)\mathfrak{m}^e_{11}(z)\mathfrak{m}^e_{21}(z)\mathrm{e}^{\mathrm{i}n\Omega^e_j} \\ \mp\frac{1}{4}(s_1-t_1)\left(\frac{(\gamma^e(z))^4-1}{(\gamma^e(z))^2}\right)\mathfrak{m}^e_{12}(z)\mathfrak{m}^e_{22}(z)\mathrm{e}^{-\mathrm{i}n\Omega^e_j} \end{matrix}} & \boxed{\begin{matrix} \pm\frac{\mathrm{i}}{2}(s_1+t_1)\left(\frac{(\gamma^e(z))^4-1}{(\gamma^e(z))^2}\right)\mathfrak{m}^e_{11}(z)\mathfrak{m}^e_{12}(z) \\ \pm\frac{\mathrm{i}}{4}(s_1-t_1)\left(\frac{(\gamma^e(z))^2+1}{\gamma^e(z)}\right)^2(\mathfrak{m}^e_{11}(z))^2\mathrm{e}^{\mathrm{i}n\Omega^e_j} \\ \pm\frac{\mathrm{i}}{4}(s_1-t_1)\left(\frac{(\gamma^e(z))^2-1}{\gamma^e(z)}\right)^2(\mathfrak{m}^e_{12}(z))^2\mathrm{e}^{-\mathrm{i}n\Omega^e_j} \end{matrix}} \\ \boxed{\begin{matrix} \pm\frac{\mathrm{i}}{2}(s_1+t_1)\left(\frac{(\gamma^e(z))^4-1}{(\gamma^e(z))^2}\right)\mathfrak{m}^e_{21}(z)\mathfrak{m}^e_{22}(z) \\ \pm\frac{\mathrm{i}}{4}(s_1-t_1)\left(\frac{(\gamma^e(z))^2-1}{\gamma^e(z)}\right)^2(\mathfrak{m}^e_{21}(z))^2\mathrm{e}^{\mathrm{i}n\Omega^e_j} \\ \pm\frac{\mathrm{i}}{4}(s_1-t_1)\left(\frac{(\gamma^e(z))^2+1}{\gamma^e(z)}\right)^2(\mathfrak{m}^e_{22}(z))^2\mathrm{e}^{-\mathrm{i}n\Omega^e_j} \end{matrix}} & \boxed{\begin{matrix} \pm\frac{1}{4}(s_1+t_1)\left(\frac{(\gamma^e(z))^2+1}{\gamma^e(z)}\right)^2\mathfrak{m}^e_{11}(z)\mathfrak{m}^e_{22}(z) \\ \pm\frac{1}{4}(s_1+t_1)\left(\frac{(\gamma^e(z))^2-1}{\gamma^e(z)}\right)^2\mathfrak{m}^e_{12}(z)\mathfrak{m}^e_{21}(z) \\ \pm\frac{1}{4}(s_1-t_1)\left(\frac{(\gamma^e(z))^4-1}{(\gamma^e(z))^2}\right)\mathfrak{m}^e_{11}(z)\mathfrak{m}^e_{21}(z)\mathrm{e}^{\mathrm{i}n\Omega^e_j} \\ \pm\frac{1}{4}(s_1-t_1)\left(\frac{(\gamma^e(z))^4-1}{(\gamma^e(z))^2}\right)\mathfrak{m}^e_{12}(z)\mathfrak{m}^e_{22}(z)\mathrm{e}^{-\mathrm{i}n\Omega^e_j} \end{matrix}} \end{pmatrix},$$

where s_1 and t_1 are given in Theorem 2.3.1, Equations (2.28), $\gamma^e(z)$ is defined in Lemma 4.4, and $\mathfrak{m}^e_{kl}(z)$, $k,l=1,2$, are defined in Theorem 2.3.1, Equations (2.17)–(2.20). Recall that, for $j=1,\dots,N$, $\omega^e_j=\sum_{k=1}^N c^e_{jk}(R_e(z))^{-1/2}z^{N-k}\,\mathrm{d}z$, where c^e_{jk}, $j,k=1,\dots,N$, are obtained from Equations (E1) and (E2), and (the multi-valued function) $(R_e(z))^{1/2}$ is defined in Theorem 2.3.1, Equation (2.8). One shows that

$$\omega^e_m \underset{\substack{z\to a^e_j\\ j=1,\dots,N}}{=} \frac{(\mathfrak{f}_e(a^e_j))^{-1}}{\sqrt{z-a^e_j}}\left(\mathfrak{p}^\natural_m(a^e_j)+\mathfrak{q}^\natural_m(a^e_j)(z-a^e_j)+\mathfrak{r}^\natural_m(a^e_j)(z-a^e_j)^2+\mathcal{O}((z-a^e_j)^3)\right)\mathrm{d}z, \quad m=1,\dots,N,$$

where

$$\mathfrak{f}_e(\xi)=(-1)^{N-j+1}\left((a_{N+1}^e-\xi)(\xi-b_0^e)(b_j^e-\xi)\prod_{k=1}^{j-1}(\xi-b_k^e)(\xi-a_k^e)\prod_{l=j+1}^{N}(b_l^e-\xi)(a_l^e-\xi)\right)^{1/2},$$

$$\mathfrak{p}_m^{\natural}(\xi)=\sum_{k=1}^{N}c_{mk}^e\xi^{N-k},\qquad \mathfrak{q}_m^{\natural}(\xi)=\sum_{k=1}^{N}c_{mk}^e\xi^{N-k-1}\left(N-k-\frac{\xi\mathfrak{f}_e'(\xi)}{\mathfrak{f}_e(\xi)}\right),$$

$$\mathfrak{r}_m^{\natural}(\xi)=\sum_{k=1}^{N}c_{mk}^e\xi^{N-k-2}\left(\frac{(N-k)(N-k-1)}{2}-\frac{(N-k)\xi\mathfrak{f}_e'(\xi)}{\mathfrak{f}_e(\xi)}+\xi^2\left(\left(\frac{\mathfrak{f}_e'(\xi)}{\mathfrak{f}_e(\xi)}\right)^2-\frac{\mathfrak{f}_e''(\xi)}{2\mathfrak{f}_e(\xi)}\right)\right),$$

with $(-1)^{-N+j-1}\mathfrak{f}_e(a_j^e)>0$,

$$\mathfrak{f}_e'(\xi)=\frac{1}{2}\mathfrak{f}_e(\xi)\left(\sum_{\substack{k=1\\k\neq j}}^{N}\left(\frac{1}{\xi-b_k^e}+\frac{1}{\xi-a_k^e}\right)+\frac{1}{\xi-b_j^e}+\frac{1}{\xi-a_{N+1}^e}+\frac{1}{\xi-b_0^e}\right),$$

$$\mathfrak{f}_e''(\xi)=-\frac{1}{2}\mathfrak{f}_e(\xi)\left(\sum_{\substack{k=1\\k\neq j}}^{N}\left(\frac{1}{(\xi-b_k^e)^2}+\frac{1}{(\xi-a_k^e)^2}\right)+\frac{1}{(\xi-b_j^e)^2}+\frac{1}{(\xi-a_{N+1}^e)^2}+\frac{1}{(\xi-b_0^e)^2}\right)$$
$$+\frac{1}{4}\mathfrak{f}_e(\xi)\left(\sum_{\substack{k=1\\k\neq j}}^{N}\left(\frac{1}{\xi-b_k^e}+\frac{1}{\xi-a_k^e}\right)+\frac{1}{\xi-b_j^e}+\frac{1}{\xi-a_{N+1}^e}+\frac{1}{\xi-b_0^e}\right)^2.$$

Recall (cf. Lemma 4.5), also, that $\boldsymbol{u}^e\equiv\int_{a_{N+1}^e}^{z}\boldsymbol{\omega}^e$ ($\in\mathrm{Jac}(\mathcal{Y}_e)$), where $\equiv$ denotes congruence modulo the period lattice, with $\boldsymbol{\omega}^e:=(\omega_1^e,\omega_2^e,\dots,\omega_N^e)$; hence, via the above expansion (as $z\to a_j^e$, $j=1,\dots,N$) for ω_m^e, $m=1,\dots,N$, one arrives at

$$\int_{a_j^e}^{z}\omega_m^e\underset{\substack{z\to a_j^e\\j=1,\dots,N}}{\equiv}\frac{2\mathfrak{p}_m^{\natural}(a_j^e)}{\mathfrak{f}_e(a_j^e)}(z-a_j^e)^{1/2}+\frac{2\mathfrak{q}_m^{\natural}(a_j^e)}{3\mathfrak{f}_e(a_j^e)}(z-a_j^e)^{3/2}+\frac{2\mathfrak{r}_m^{\natural}(a_j^e)}{5\mathfrak{f}_e(a_j^e)}(z-a_j^e)^{5/2}+O((z-a_j^e)^{7/2}).$$

From the definition of $\mathfrak{m}_{kl}^e(z)$, $k,l=1,2$, given in Theorem 2.3.1, Equations (2.17)–(2.20), the definition of the 'even' Riemann theta function given by Equation (2.1), and recalling that $\mathfrak{m}_{kl}^e(z)$, $k,l=1,2$, satisfy the jump relation (cf. Lemma 4.5) $\mathfrak{m}_+^e(z)=\mathfrak{m}_-^e(z)(\exp(-\mathrm{i}n\Omega_j^e)\sigma_-+\exp(\mathrm{i}n\Omega_j^e)\sigma_+)$, via the above asymptotic expansion (as $z\to a_j^e$, $j=1,\dots,N$) for $\int_{a_j^e}^{z}\omega_m^e$, $m=1,\dots,N$, one arrives at

$$\mathfrak{m}_{11}^e(z)\underset{\substack{z\to a_j^e\\j=1,\dots,N}}{=}\varkappa_1^e(a_j^e)\Big(1+\mathrm{i}\aleph_1^1(a_j^e)(z-a_j^e)^{1/2}+\daleth_1^1(a_j^e)(z-a_j^e)+\mathrm{i}\beth_1^1(a_j^e)(z-a_j^e)^{3/2}+\gimel_1^1(a_j^e)(z-a_j^e)^2$$
$$+\,O((z-a_j^e)^{5/2})\Big),$$

$$\mathfrak{m}_{12}^e(z)\underset{\substack{z\to a_j^e\\j=1,\dots,N}}{=}\varkappa_1^e(a_j^e)\Big(1-\mathrm{i}\aleph_1^{-1}(a_j^e)(z-a_j^e)^{1/2}+\daleth_1^{-1}(a_j^e)(z-a_j^e)-\mathrm{i}\beth_1^{-1}(a_j^e)(z-a_j^e)^{3/2}+\gimel_1^{-1}(a_j^e)(z-a_j^e)^2$$
$$+\,O((z-a_j^e)^{5/2})\Big)\exp\big(\mathrm{i}n\Omega_j^e\big),$$

$$\mathfrak{m}_{21}^e(z)\underset{\substack{z\to a_j^e\\j=1,\dots,N}}{=}\varkappa_2^e(a_j^e)\Big(1+\mathrm{i}\aleph_{-1}^1(a_j^e)(z-a_j^e)^{1/2}+\daleth_{-1}^1(a_j^e)(z-a_j^e)+\mathrm{i}\beth_{-1}^1(a_j^e)(z-a_j^e)^{3/2}+\gimel_{-1}^1(a_j^e)(z-a_j^e)^2$$
$$+\,O((z-a_j^e)^{5/2})\Big),$$

$$\mathfrak{m}_{22}^e(z)\underset{\substack{z\to a_j^e\\j=1,\dots,N}}{=}\varkappa_2^e(a_j^e)\Big(1-\mathrm{i}\aleph_{-1}^{-1}(a_j^e)(z-a_j^e)^{1/2}+\daleth_{-1}^{-1}(a_j^e)(z-a_j^e)-\mathrm{i}\beth_{-1}^{-1}(a_j^e)(z-a_j^e)^{3/2}+\gimel_{-1}^{-1}(a_j^e)(z-a_j^e)^2$$
$$+\,O((z-a_j^e)^{5/2})\Big)\exp\big(\mathrm{i}n\Omega_j^e\big),$$

where, for $\varepsilon_1,\varepsilon_2=\pm 1$,

$$
\begin{aligned}
\varkappa^{e}_{1}(\xi) &= \frac{\boldsymbol{\theta}^{e}(\boldsymbol{u}^{e}_{+}(\infty)+\boldsymbol{d}_e)\boldsymbol{\theta}^{e}(\boldsymbol{u}^{e}_{+}(\xi)-\frac{n}{2\pi}\boldsymbol{\Omega}^{e}+\boldsymbol{d}_e)}{\boldsymbol{\theta}^{e}(\boldsymbol{u}^{e}_{+}(\infty)-\frac{n}{2\pi}\boldsymbol{\Omega}^{e}+\boldsymbol{d}_e)\boldsymbol{\theta}^{e}(\boldsymbol{u}^{e}_{+}(\xi)+\boldsymbol{d}_e)},\\
\varkappa^{e}_{2}(\xi) &= \frac{\boldsymbol{\theta}^{e}(-\boldsymbol{u}^{e}_{+}(\infty)-\boldsymbol{d}_e)\boldsymbol{\theta}^{e}(\boldsymbol{u}^{e}_{+}(\xi)-\frac{n}{2\pi}\boldsymbol{\Omega}^{e}-\boldsymbol{d}_e)}{\boldsymbol{\theta}^{e}(-\boldsymbol{u}^{e}_{+}(\infty)-\frac{n}{2\pi}\boldsymbol{\Omega}^{e}-\boldsymbol{d}_e)\boldsymbol{\theta}^{e}(\boldsymbol{u}^{e}_{+}(\xi)-\boldsymbol{d}_e)},\\
\aleph^{\varepsilon_1}_{\varepsilon_2}(\xi) &= -\frac{\mathfrak{u}^{e}(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi)}{\boldsymbol{\theta}^{e}(\varepsilon_1\boldsymbol{u}^{e}_{+}(\xi)+\varepsilon_2\boldsymbol{d}_e)}+\frac{\mathfrak{u}^{e}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{e};\xi)}{\boldsymbol{\theta}^{e}(\varepsilon_1\boldsymbol{u}^{e}_{+}(\xi)-\frac{n}{2\pi}\boldsymbol{\Omega}^{e}+\varepsilon_2\boldsymbol{d}_e)},\\
\daleth^{\varepsilon_1}_{\varepsilon_2}(\xi) &= -\frac{\mathfrak{v}^{e}(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi)}{\boldsymbol{\theta}^{e}(\varepsilon_1\boldsymbol{u}^{e}_{+}(\xi)+\varepsilon_2\boldsymbol{d}_e)}+\frac{\mathfrak{v}^{e}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{e};\xi)}{\boldsymbol{\theta}^{e}(\varepsilon_1\boldsymbol{u}^{e}_{+}(\xi)-\frac{n}{2\pi}\boldsymbol{\Omega}^{e}+\varepsilon_2\boldsymbol{d}_e)}-\left(\frac{\mathfrak{u}^{e}(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi)}{\boldsymbol{\theta}^{e}(\varepsilon_1\boldsymbol{u}^{e}_{+}(\xi)+\varepsilon_2\boldsymbol{d}_e)}\right)^{2}\\
&\quad+\frac{\mathfrak{u}^{e}(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi)\mathfrak{u}^{e}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{e};\xi)}{\boldsymbol{\theta}^{e}(\varepsilon_1\boldsymbol{u}^{e}_{+}(\xi)+\varepsilon_2\boldsymbol{d}_e)\boldsymbol{\theta}^{e}(\varepsilon_1\boldsymbol{u}^{e}_{+}(\xi)-\frac{n}{2\pi}\boldsymbol{\Omega}^{e}+\varepsilon_2\boldsymbol{d}_e)},\\
\beth^{\varepsilon_1}_{\varepsilon_2}(\xi) &= -\frac{\mathfrak{w}^{e}(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi)}{\boldsymbol{\theta}^{e}(\varepsilon_1\boldsymbol{u}^{e}_{+}(\xi)+\varepsilon_2\boldsymbol{d}_e)}+\frac{\mathfrak{w}^{e}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{e};\xi)}{\boldsymbol{\theta}^{e}(\varepsilon_1\boldsymbol{u}^{e}_{+}(\xi)-\frac{n}{2\pi}\boldsymbol{\Omega}^{e}+\varepsilon_2\boldsymbol{d}_e)}+\frac{2\mathfrak{u}^{e}(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi)\mathfrak{v}^{e}(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi)}{(\boldsymbol{\theta}^{e}(\varepsilon_1\boldsymbol{u}^{e}_{+}(\xi)+\varepsilon_2\boldsymbol{d}_e))^{2}}\\
&\quad-\frac{\mathfrak{v}^{e}(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi)\mathfrak{u}^{e}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{e};\xi)}{\boldsymbol{\theta}^{e}(\varepsilon_1\boldsymbol{u}^{e}_{+}(\xi)+\varepsilon_2\boldsymbol{d}_e)\boldsymbol{\theta}^{e}(\varepsilon_1\boldsymbol{u}^{e}_{+}(\xi)-\frac{n}{2\pi}\boldsymbol{\Omega}^{e}+\varepsilon_2\boldsymbol{d}_e)}+\left(\frac{\mathfrak{u}^{e}(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi)}{\boldsymbol{\theta}^{e}(\varepsilon_1\boldsymbol{u}^{e}_{+}(\xi)+\varepsilon_2\boldsymbol{d}_e)}\right)^{3}\\
&\quad-\frac{\mathfrak{u}^{e}(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi)\mathfrak{v}^{e}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{e};\xi)}{\boldsymbol{\theta}^{e}(\varepsilon_1\boldsymbol{u}^{e}_{+}(\xi)+\varepsilon_2\boldsymbol{d}_e)\boldsymbol{\theta}^{e}(\varepsilon_1\boldsymbol{u}^{e}_{+}(\xi)-\frac{n}{2\pi}\boldsymbol{\Omega}^{e}+\varepsilon_2\boldsymbol{d}_e)}-\frac{(\mathfrak{u}^{e}(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi))^{2}}{(\boldsymbol{\theta}^{e}(\varepsilon_1\boldsymbol{u}^{e}_{+}(\xi)+\varepsilon_2\boldsymbol{d}_e))^{2}}\\
&\quad\times\frac{\mathfrak{u}^{e}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{e};\xi)}{\boldsymbol{\theta}^{e}(\varepsilon_1\boldsymbol{u}^{e}_{+}(\xi)-\frac{n}{2\pi}\boldsymbol{\Omega}^{e}+\varepsilon_2\boldsymbol{d}_e)},\\
\gimel^{\varepsilon_1}_{\varepsilon_2}(\xi) &= -\frac{\mathfrak{z}^{e}(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi)}{\boldsymbol{\theta}^{e}(\varepsilon_1\boldsymbol{u}^{e}_{+}(\xi)+\varepsilon_2\boldsymbol{d}_e)}+\frac{\mathfrak{z}^{e}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{e};\xi)}{\boldsymbol{\theta}^{e}(\varepsilon_1\boldsymbol{u}^{e}_{+}(\xi)-\frac{n}{2\pi}\boldsymbol{\Omega}^{e}+\varepsilon_2\boldsymbol{d}_e)}+\left(\frac{\mathfrak{v}^{e}(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi)}{\boldsymbol{\theta}^{e}(\varepsilon_1\boldsymbol{u}^{e}_{+}(\xi)+\varepsilon_2\boldsymbol{d}_e)}\right)^{2}\\
&\quad-\frac{\mathfrak{v}^{e}(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi)\mathfrak{v}^{e}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{e};\xi)}{\boldsymbol{\theta}^{e}(\varepsilon_1\boldsymbol{u}^{e}_{+}(\xi)+\varepsilon_2\boldsymbol{d}_e)\boldsymbol{\theta}^{e}(\varepsilon_1\boldsymbol{u}^{e}_{+}(\xi)-\frac{n}{2\pi}\boldsymbol{\Omega}^{e}+\varepsilon_2\boldsymbol{d}_e)}-\frac{2\mathfrak{u}^{e}(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi)\mathfrak{w}^{e}(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi)}{(\boldsymbol{\theta}^{e}(\varepsilon_1\boldsymbol{u}^{e}_{+}(\xi)+\varepsilon_2\boldsymbol{d}_e))^{2}}\\
&\quad+\frac{\mathfrak{w}^{e}(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi)\mathfrak{u}^{e}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{e};\xi)}{\boldsymbol{\theta}^{e}(\varepsilon_1\boldsymbol{u}^{e}_{+}(\xi)+\varepsilon_2\boldsymbol{d}_e)\boldsymbol{\theta}^{e}(\varepsilon_1\boldsymbol{u}^{e}_{+}(\xi)-\frac{n}{2\pi}\boldsymbol{\Omega}^{e}+\varepsilon_2\boldsymbol{d}_e)}+\frac{3(\mathfrak{u}^{e}(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi))^{2}\mathfrak{v}^{e}(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi)}{(\boldsymbol{\theta}^{e}(\varepsilon_1\boldsymbol{u}^{e}_{+}(\xi)+\varepsilon_2\boldsymbol{d}_e))^{3}}\\
&\quad+\frac{\mathfrak{u}^{e}(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi)\mathfrak{w}^{e}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{e};\xi)}{\boldsymbol{\theta}^{e}(\varepsilon_1\boldsymbol{u}^{e}_{+}(\xi)+\varepsilon_2\boldsymbol{d}_e)\boldsymbol{\theta}^{e}(\varepsilon_1\boldsymbol{u}^{e}_{+}(\xi)-\frac{n}{2\pi}\boldsymbol{\Omega}^{e}+\varepsilon_2\boldsymbol{d}_e)}+\left(\frac{\mathfrak{u}^{e}(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi)}{\boldsymbol{\theta}^{e}(\varepsilon_1\boldsymbol{u}^{e}_{+}(\xi)+\varepsilon_2\boldsymbol{d}_e)}\right)^{4}\\
&\quad-\frac{2\mathfrak{u}^{e}(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi)\mathfrak{v}^{e}(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi)\mathfrak{u}^{e}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{e};\xi)}{(\boldsymbol{\theta}^{e}(\varepsilon_1\boldsymbol{u}^{e}_{+}(\xi)+\varepsilon_2\boldsymbol{d}_e))^{2}\boldsymbol{\theta}^{e}(\varepsilon_1\boldsymbol{u}^{e}_{+}(\xi)-\frac{n}{2\pi}\boldsymbol{\Omega}^{e}+\varepsilon_2\boldsymbol{d}_e)}-\frac{(\mathfrak{u}^{e}(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi))^{2}}{(\boldsymbol{\theta}^{e}(\varepsilon_1\boldsymbol{u}^{e}_{+}(\xi)+\varepsilon_2\boldsymbol{d}_e))^{2}}\\
&\quad\times\frac{\mathfrak{v}^{e}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{e};\xi)}{\boldsymbol{\theta}^{e}(\varepsilon_1\boldsymbol{u}^{e}_{+}(\xi)-\frac{n}{2\pi}\boldsymbol{\Omega}^{e}+\varepsilon_2\boldsymbol{d}_e)}-\frac{(\mathfrak{u}^{e}(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi))^{3}\mathfrak{u}^{e}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{e};\xi)}{(\boldsymbol{\theta}^{e}(\varepsilon_1\boldsymbol{u}^{e}_{+}(\xi)+\varepsilon_2\boldsymbol{d}_e))^{3}\boldsymbol{\theta}^{e}(\varepsilon_1\boldsymbol{u}^{e}_{+}(\xi)-\frac{n}{2\pi}\boldsymbol{\Omega}^{e}+\varepsilon_2\boldsymbol{d}_e)},
\end{aligned}
$$

with $\mathbf{0}:=(0,0,\dots,0)^{\mathrm{T}}$ $(\in\mathbb{R}^{N})$,

$$
\mathfrak{u}^{e}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{e};\xi):=2\pi\overset{e}{\Lambda}{}^{1}_{0}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{e};\xi),\qquad \mathfrak{v}^{e}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{e};\xi):=-2\pi^{2}\overset{e}{\Lambda}{}^{2}_{0}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{e};\xi),
$$

$$
\mathfrak{w}^{e}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{e};\xi):=2\pi\left(\overset{e}{\Lambda}{}^{0}_{1}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{e};\xi)-\frac{2\pi^{2}}{3}\overset{e}{\Lambda}{}^{3}_{0}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{e};\xi)\right),
$$

$$
\mathfrak{z}^{e}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{e};\xi):=-(2\pi)^{2}\left(\overset{e}{\Lambda}{}^{1}_{1}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{e};\xi)-\frac{\pi^{2}}{6}\overset{e}{\Lambda}{}^{4}_{0}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{e};\xi)\right),
$$

$$
\overset{e}{\Lambda}{}^{j_1}_{j_2}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{e};\xi)=\sum_{m\in\mathbb{Z}^{N}}(\mathfrak{r}_e(\xi))^{j_1}(\mathfrak{s}_e(\xi))^{j_2}\,\mathrm{e}^{2\pi\mathrm{i}(m,\varepsilon_1\boldsymbol{u}^{e}_{+}(\xi)-\frac{n}{2\pi}\boldsymbol{\Omega}^{e}+\varepsilon_2\boldsymbol{d}_e)+\pi\mathrm{i}(m,\boldsymbol{\tau}^{e}m)},
$$

$$
\mathfrak{r}_e(\xi):=\frac{2(m,\vec{\mathfrak{X}}^{e}_{1}(\xi))}{\mathfrak{f}_e(\xi)},\qquad \mathfrak{s}_e(\xi):=\frac{2(m,\vec{\mathfrak{X}}^{e}_{2}(\xi))}{3\mathfrak{f}_e(\xi)},
$$

$$
\vec{\mathfrak{X}}^{e}_{1}(\xi)=\left(\mathfrak{p}^{\natural}_{1}(\xi),\mathfrak{p}^{\natural}_{2}(\xi),\dots,\mathfrak{p}^{\natural}_{N}(\xi)\right),\qquad \vec{\mathfrak{X}}^{e}_{2}(\xi)=\left(\mathfrak{q}^{\natural}_{1}(\xi),\mathfrak{q}^{\natural}_{2}(\xi),\dots,\mathfrak{q}^{\natural}_{N}(\xi)\right).
$$

Recall the definition of $\gamma^{e}(z)$ given in Lemma 4.4: a careful analysis of the branch cuts shows that,

for $j=1,\dots,N$,

$$(\gamma^{e}(z))^{2} \underset{z\in\mathbb{C}_{\pm}}{=} \pm\frac{\left((z-b_{j}^{e})\prod_{\substack{k=1\\k\neq j}}^{N}\left(\frac{z-b_{k}^{e}}{z-a_{k}^{e}}\right)\left(\frac{z-b_{0}^{e}}{z-a_{N+1}^{e}}\right)\right)^{1/2}}{\sqrt{z-a_{j}^{e}}}$$

$$\underset{\mathbb{C}_{\pm}\ni z\to a_{j}^{e}}{=} \pm\frac{\left(Q_{0}^{e}(a_{j}^{e})+Q_{1}^{e}(a_{j}^{e})(z-a_{j}^{e})+\frac{1}{2}Q_{2}^{e}(a_{j}^{e})(z-a_{j}^{e})^{2}+\mathcal{O}((z-a_{j}^{e})^{3})\right)}{\sqrt{z-a_{j}^{e}}},$$

where $Q_{0}^{e}(a_{j}^{e}), Q_{1}^{e}(a_{j}^{e})$, $j=1,\dots,N$, are given in Theorem 2.3.1, Equations (2.35) and (2.36), and

$$Q_{2}^{e}(a_{j}^{e}) = -\frac{1}{2}Q_{0}^{e}(a_{j}^{e})\left(\sum_{\substack{k=1\\k\neq j}}^{N}\left(\frac{1}{(a_{j}^{e}-b_{k}^{e})^{2}}-\frac{1}{(a_{j}^{e}-a_{k}^{e})^{2}}\right)+\frac{1}{(a_{j}^{e}-b_{0}^{e})^{2}}-\frac{1}{(a_{j}^{e}-a_{N+1}^{e})^{2}}+\frac{1}{(a_{j}^{e}-b_{j}^{e})^{2}}\right)$$
$$+\frac{1}{4}Q_{0}^{e}(a_{j}^{e})\left(\sum_{\substack{k=1\\k\neq j}}^{N}\left(\frac{1}{a_{j}^{e}-b_{k}^{e}}-\frac{1}{a_{j}^{e}-a_{k}^{e}}\right)+\frac{1}{a_{j}^{e}-b_{0}^{e}}-\frac{1}{a_{j}^{e}-a_{N+1}^{e}}+\frac{1}{a_{j}^{e}-b_{j}^{e}}\right)^{2}, \quad j=1,\dots,N.$$

Recall the above formula for $\overset{e}{\mathfrak{M}}^{\infty}(z)\begin{pmatrix} \mp(s_1+t_1) & \pm\mathrm{i}(s_1-t_1)\mathrm{e}^{\mathrm{i}n\Omega_{j}^{e}} \\ \pm\mathrm{i}(s_1-t_1)\mathrm{e}^{-\mathrm{i}n\Omega_{j}^{e}} & \pm(s_1+t_1)\end{pmatrix}(\overset{e}{\mathfrak{M}}^{\infty}(z))^{-1}$. Substituting the above expansions (as $z\to a_{j}^{e}$, $j=1,\dots,N$) into this formula, equating coefficients of like powers of $(z-a_{j}^{e})^{-p/2}(nG_{a_j}^{e}(z))^{-1}$, $p\in\{4,3,2,1,0\}$, and considering, say, the $(1\,1)$-element of the resulting (asymptotic) expansions, one arrives at, up to terms that are $\mathcal{O}\Big((n^{2}(z-a_{j}^{e})^{3}(G_{a_j}^{e}(z))^{2})^{-1}\overset{e}{\mathfrak{M}}^{\infty}(z)f_{a_j}^{e}(n)(\overset{e}{\mathfrak{M}}^{\infty}(z))^{-1}\Big)$ (modulo a minus sign, this result is equally applicable to the $(2\,2)$-element, since $\operatorname{tr}(w_{+}^{\Sigma_{\circlearrowright}^{e}}(z))=0$), upon setting, for economy of notation, $Q_{q}^{e}(a_{j}^{e})=:Q_{q}$, $q=0,1,2$, $\varkappa_{1}^{e}(a_{j}^{e})=:\varkappa_{1}^{e}$, $\varkappa_{2}^{e}(a_{j}^{e})=:\varkappa_{2}^{e}$, $\aleph_{\varepsilon_2}^{\varepsilon_1}(a_{j}^{e})=:\aleph_{\varepsilon_2}^{\varepsilon_1}$, $\daleth_{\varepsilon_2}^{\varepsilon_1}(a_{j}^{e})=:\daleth_{\varepsilon_2}^{\varepsilon_1}$, $\beth_{\varepsilon_2}^{\varepsilon_1}(a_{j}^{e})=:\beth_{\varepsilon_2}^{\varepsilon_1}$, and $\gimel_{\varepsilon_2}^{\varepsilon_1}(a_{j}^{e})=:\gimel_{\varepsilon_2}^{\varepsilon_1}$:

$$\mathcal{O}\left(\frac{(z-a_{j}^{e})^{-2}\,\mathrm{e}^{\mathrm{i}n\Omega_{j}^{e}}}{nG_{a_j}^{e}(z)}\right): -\frac{(s_1+t_1)\varkappa_{1}^{e}\varkappa_{2}^{e}Q_0}{4}-\frac{(s_1+t_1)\varkappa_{1}^{e}\varkappa_{2}^{e}Q_0}{4}-\frac{(s_1-t_1)\varkappa_{1}^{e}\varkappa_{2}^{e}Q_0}{4}-\frac{(s_1-t_1)\varkappa_{1}^{e}\varkappa_{2}^{e}Q_0}{4};$$

$$\mathcal{O}\left(\frac{(z-a_{j}^{e})^{-3/2}\,\mathrm{e}^{\mathrm{i}n\Omega_{j}^{e}}}{nG_{a_j}^{e}(z)}\right): -\frac{\mathrm{i}(s_1+t_1)\varkappa_{1}^{e}\varkappa_{2}^{e}Q_0(\aleph_{1}^{1}-\aleph_{-1}^{-1})}{4}-\frac{\mathrm{i}(s_1+t_1)\varkappa_{1}^{e}\varkappa_{2}^{e}Q_0(\aleph_{-1}^{1}-\aleph_{1}^{-1})}{4}$$
$$-\frac{\mathrm{i}(s_1-t_1)\varkappa_{1}^{e}\varkappa_{2}^{e}Q_0(\aleph_{-1}^{1}+\aleph_{1}^{1})}{4}+\frac{\mathrm{i}(s_1-t_1)\varkappa_{1}^{e}\varkappa_{2}^{e}Q_0(\aleph_{-1}^{-1}+\aleph_{1}^{-1})}{4}$$
$$-\frac{(s_1+t_1)\varkappa_{1}^{e}\varkappa_{2}^{e}}{2}+\frac{(s_1+t_1)\varkappa_{1}^{e}\varkappa_{2}^{e}}{2};$$

$$\mathcal{O}\left(\frac{(z-a_{j}^{e})^{-1}\,\mathrm{e}^{\mathrm{i}n\Omega_{j}^{e}}}{nG_{a_j}^{e}(z)}\right): -\frac{(s_1+t_1)\varkappa_{1}^{e}\varkappa_{2}^{e}}{4}\Big(Q_1+Q_0\Big(\daleth_{-1}^{-1}+\daleth_{1}^{1}+\aleph_{1}^{1}\aleph_{-1}^{-1}\Big)\Big)-\frac{(s_1+t_1)\varkappa_{1}^{e}\varkappa_{2}^{e}}{4Q_0}$$
$$-\frac{(s_1+t_1)\varkappa_{1}^{e}\varkappa_{2}^{e}}{4}\Big(Q_1+Q_0\Big(\daleth_{-1}^{1}+\daleth_{1}^{-1}+\aleph_{1}^{-1}\aleph_{-1}^{1}\Big)\Big)-\frac{(s_1+t_1)\varkappa_{1}^{e}\varkappa_{2}^{e}}{4Q_0}$$
$$-\frac{(s_1-t_1)\varkappa_{1}^{e}\varkappa_{2}^{e}}{4}\Big(Q_1+Q_0\Big(\daleth_{-1}^{1}+\daleth_{1}^{1}-\aleph_{1}^{1}\aleph_{-1}^{1}\Big)\Big)+\frac{(s_1-t_1)\varkappa_{1}^{e}\varkappa_{2}^{e}}{4Q_0}$$
$$-\frac{(s_1-t_1)\varkappa_{1}^{e}\varkappa_{2}^{e}}{4}\Big(Q_1+Q_0\Big(\daleth_{-1}^{-1}+\daleth_{1}^{-1}-\aleph_{1}^{-1}\aleph_{-1}^{-1}\Big)\Big)+\frac{(s_1-t_1)\varkappa_{1}^{e}\varkappa_{2}^{e}}{4Q_0}$$
$$-\frac{\mathrm{i}(s_1+t_1)\varkappa_{1}^{e}\varkappa_{2}^{e}}{2}\Big(\aleph_{1}^{1}-\aleph_{-1}^{-1}\Big)+\frac{\mathrm{i}(s_1+t_1)\varkappa_{1}^{e}\varkappa_{2}^{e}}{2}\Big(\aleph_{-1}^{1}-\aleph_{1}^{-1}\Big);$$

$$\mathcal{O}\left(\frac{(z-a_{j}^{e})^{-1/2}\,\mathrm{e}^{\mathrm{i}n\Omega_{j}^{e}}}{nG_{a_j}^{e}(z)}\right): -\frac{\mathrm{i}(s_1+t_1)\varkappa_{1}^{e}\varkappa_{2}^{e}}{4}\Big(Q_1\Big(\aleph_{1}^{1}-\aleph_{-1}^{-1}\Big)+Q_0\Big(\beth_{1}^{1}-\beth_{-1}^{-1}+\aleph_{1}^{1}\daleth_{-1}^{-1}-\aleph_{-1}^{-1}\daleth_{1}^{1}\Big)\Big)$$

$$
\begin{aligned}
&-\frac{\mathrm{i}(s_1+t_1)\varkappa_1^e\varkappa_2^e}{4}\Big(Q_1\big(\aleph_{-1}^1-\aleph_1^{-1}\big)+Q_0\big(\beth_{-1}^1-\beth_1^{-1}+\aleph_{-1}^1\daleth_1^{-1}-\aleph_1^{-1}\daleth_{-1}^1\big)\Big)\\
&-\frac{\mathrm{i}(s_1-t_1)\varkappa_1^e\varkappa_2^e}{4}\Big(Q_1\big(\aleph_{-1}^1+\aleph_1^1\big)+Q_0\big(\beth_{-1}^1+\beth_1^1+\aleph_1^1\daleth_{-1}^1+\aleph_{-1}^1\daleth_1^1\big)\Big)\\
&+\frac{\mathrm{i}(s_1-t_1)\varkappa_1^e\varkappa_2^e}{4}\Big(Q_1\big(\aleph_{-1}^{-1}+\aleph_1^{-1}\big)+Q_0\big(\beth_{-1}^{-1}+\beth_1^{-1}+\aleph_1^{-1}\daleth_{-1}^{-1}+\aleph_{-1}^{-1}\daleth_1^{-1}\big)\Big)\\
&-\frac{\mathrm{i}(s_1+t_1)\varkappa_1^e\varkappa_2^e}{4Q_0}\big(\aleph_1^1-\aleph_{-1}^{-1}\big)-\frac{(s_1+t_1)\varkappa_1^e\varkappa_2^e}{2}\big(\daleth_{-1}^{-1}+\daleth_1^1+\aleph_1^1\aleph_{-1}^{-1}\big)\\
&-\frac{\mathrm{i}(s_1+t_1)\varkappa_1^e\varkappa_2^e}{4Q_0}\big(\aleph_{-1}^1-\aleph_1^{-1}\big)+\frac{(s_1+t_1)\varkappa_1^e\varkappa_2^e}{2}\big(\daleth_{-1}^1+\daleth_1^{-1}+\aleph_1^{-1}\aleph_{-1}^1\big)\\
&+\frac{\mathrm{i}(s_1-t_1)\varkappa_1^e\varkappa_2^e}{4Q_0}\big(\aleph_{-1}^1+\aleph_1^1\big)-\frac{\mathrm{i}(s_1-t_1)\varkappa_1^e\varkappa_2^e}{4Q_0}\big(\aleph_1^{-1}+\aleph_{-1}^{-1}\big);
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{O}\left(\frac{\mathrm{e}^{\mathrm{i}n\Omega_j^e}}{nG_{a_j}^e(z)}\right):&-\frac{(s_1+t_1)\varkappa_1^e\varkappa_2^e}{4}\Big(Q_0\big(\gimel_{-1}^{-1}+\gimel_1^1+\daleth_1^1\daleth_{-1}^{-1}+\aleph_{-1}^{-1}\beth_1^1+\aleph_1^1\beth_{-1}^{-1}\big)+Q_1\big(\daleth_{-1}^{-1}+\daleth_1^1+\aleph_1^1\aleph_{-1}^{-1}\big)\\
&+\frac{1}{2}Q_2\Big)-\frac{(s_1+t_1)\varkappa_1^e\varkappa_2^e}{4}\Big(Q_0\big(\gimel_{-1}^1+\gimel_1^{-1}+\daleth_1^{-1}\daleth_{-1}^1+\aleph_1^{-1}\beth_{-1}^1+\aleph_{-1}^1\beth_1^{-1}\big)+\frac{1}{2}Q_2\\
&+Q_1\big(\daleth_{-1}^1+\daleth_1^{-1}+\aleph_1^{-1}\aleph_{-1}^1\big)\Big)-\frac{(s_1-t_1)\varkappa_1^e\varkappa_2^e}{4}\Big(Q_0\big(\gimel_{-1}^1+\gimel_1^1+\daleth_1^1\daleth_{-1}^1-\aleph_1^1\beth_{-1}^1-\aleph_{-1}^1\beth_1^1\big)\\
&+\frac{1}{2}Q_2+Q_1\big(\daleth_{-1}^1+\daleth_1^1-\aleph_1^1\aleph_{-1}^1\big)\Big)-\frac{(s_1-t_1)\varkappa_1^e\varkappa_2^e}{4}\Big(Q_0\big(\gimel_{-1}^{-1}+\gimel_1^{-1}+\daleth_1^{-1}\daleth_{-1}^{-1}-\aleph_1^{-1}\beth_{-1}^{-1}\\
&-\aleph_{-1}^{-1}\beth_1^{-1}\big)+\frac{1}{2}Q_2+Q_1\big(\daleth_{-1}^{-1}+\daleth_1^{-1}-\aleph_1^{-1}\aleph_{-1}^{-1}\big)\Big)-\frac{(s_1+t_1)\varkappa_1^e\varkappa_2^e}{4Q_0}\Big(\daleth_{-1}^{-1}+\daleth_1^1+\aleph_1^1\aleph_{-1}^{-1}\\
&-Q_1(Q_0)^{-1}\Big)-\frac{(s_1+t_1)\varkappa_1^e\varkappa_2^e}{4Q_0}\Big(\daleth_{-1}^1+\daleth_1^{-1}+\aleph_1^{-1}\aleph_{-1}^1-Q_1(Q_0)^{-1}\Big)-\frac{\mathrm{i}(s_1+t_1)\varkappa_1^e\varkappa_2^e}{2}\\
&\times\big(\beth_1^1-\beth_{-1}^{-1}+\aleph_1^1\daleth_{-1}^{-1}-\aleph_{-1}^{-1}\daleth_1^1\big)+\frac{\mathrm{i}(s_1+t_1)\varkappa_1^e\varkappa_2^e}{2}\big(\beth_{-1}^1-\beth_1^{-1}+\aleph_{-1}^1\daleth_1^{-1}-\aleph_1^{-1}\daleth_{-1}^1\big)\\
&+\frac{(s_1-t_1)\varkappa_1^e\varkappa_2^e}{4Q_0}\Big(\daleth_{-1}^1+\daleth_1^1-\aleph_1^1\aleph_{-1}^1-Q_1(Q_0)^{-1}\Big)+\frac{(s_1-t_1)\varkappa_1^e\varkappa_2^e}{4Q_0}\Big(\daleth_{-1}^{-1}+\daleth_1^{-1}\\
&-\aleph_1^{-1}\aleph_{-1}^{-1}-Q_1(Q_0)^{-1}\Big).
\end{aligned}
$$

Repeating the above analysis, *mutatis mutandis*, for the (1 2)- and (2 1)-elements, substituting $\aleph_1^{-1}=\aleph_1^1$, $\aleph_{-1}^{-1}=\aleph_{-1}^1$, $\daleth_1^{-1}=\daleth_1^1$, $\daleth_{-1}^{-1}=\daleth_{-1}^1$, $\beth_1^{-1}=\beth_1^1$, $\beth_{-1}^{-1}=\beth_{-1}^1$, $\gimel_1^{-1}=\gimel_1^1$, and $\gimel_{-1}^{-1}=\gimel_{-1}^1$ into the above (and resulting) 'coefficient equations', and simplifying, one shows that: (i) the coefficients of the terms that are $\mathcal{O}((z-a_j^e)^{-p/2}(nG_{a_j}^e(z))^{-1}\exp(\mathrm{i}n\Omega_j^e))$, $p=1,3$, are equal to zero; and (ii) recalling from Lemma 4.7 that, for $z\in\mathbb{U}_{\delta_{a_j}}^e\setminus(-\infty,a_j^e)$, $j=1,\dots,N$, $G_{a_j}^e(z)=_{z\to a_j^e}\widehat{\alpha}_0+\widehat{\alpha}_1(z-a_j^e)+\widehat{\alpha}_2(z-a_j^e)^2+\mathcal{O}((z-a_j^e)^3)$, where $\widehat{\alpha}_0=\widehat{\alpha}_0^e(a_j^e):=\frac{4}{3}f(a_j^e)$, $\widehat{\alpha}_1=\widehat{\alpha}_1^e(a_j^e):=\frac{4}{5}f'(a_j^e)$, and $\widehat{\alpha}_2=\widehat{\alpha}_2^e(a_j^e):=\frac{2}{7}f''(a_j^e)$, with $f(a_j^e)$, $f'(a_j^e)$, and $f''(a_j^e)$ given in Lemma 4.7, substituting the expansion for $G_{a_j}^e(z)$ (as $z\to a_j^e$, $j=1,\dots,N$) into the remaining non-zero coefficient equations, collecting coefficients of like powers of $(z-a_j^e)^{-p}$, $p=0,1,2$, and continuing, analytically, the resulting (rational) expressions to $\partial\mathbb{U}_{\delta_{a_j}}^e$, $j=1,\dots,N$, one arrives at, after a lengthy calculation and reinserting explicit a_j^e, $j=1,\dots,N$, dependencies,

$$
\begin{aligned}
w_+^{\Sigma_\circlearrowright^e}(z) \underset{\substack{n\to\infty\\ z\in\partial\mathbb{U}_{\delta_{a_j}}^e}}{=} \frac{1}{n}\Bigg(&\frac{\mathcal{A}^e(a_j^e)}{\widehat{\alpha}_0^e(a_j^e)(z-a_j^e)^2}+\frac{(\mathcal{B}^e(a_j^e)\widehat{\alpha}_0^e(a_j^e)-\mathcal{A}^e(a_j^e)\widehat{\alpha}_1^e(a_j^e))}{(\widehat{\alpha}_0^e(a_j^e))^2(z-a_j^e)}\\
&+\frac{\left(\mathcal{A}^e(a_j^e)\widehat{\alpha}_0^e(a_j^e)\left(\left(\frac{\widehat{\alpha}_1^e(a_j^e)}{\widehat{\alpha}_0^e(a_j^e)}\right)^2-\frac{\widehat{\alpha}_2^e(a_j^e)}{\widehat{\alpha}_0^e(a_j^e)}\right)-\mathcal{B}^e(a_j^e)\widehat{\alpha}_1^e(a_j^e)+\mathcal{C}^e(a_j^e)\widehat{\alpha}_0^e(a_j^e)\right)}{(\widehat{\alpha}_0^e(a_j^e))^2}\Bigg)
\end{aligned}
$$

$$+\,\mathcal{O}\left(\frac{1}{n}\sum_{k=1}^{\infty} f_k^e(n)(z-a_j^e)^k\right)+\mathcal{O}\left(\frac{\overset{e}{\mathfrak{M}}{}^{\infty}(z) f^e_{a_j}(n)(\overset{e}{\mathfrak{M}}{}^{\infty}(z))^{-1}}{n^2(z-a_j^e)^3(G^e_{a_j}(z))^2}\right), \tag{5.3}$$

where $\mathcal{A}^e(a_j^e)$, $\mathcal{B}^e(a_j^e)$, $j=1,\dots,N$, are defined in Theorem 2.3.1, Equations (2.25), (2.27), (2.28), (2.35)–(2.45), (2.49), (2.56), and (2.57),

$$\frac{\mathcal{C}^e(a_j^e)}{\mathrm{e}^{\mathrm{i}n\Omega_j^e}} := \begin{pmatrix}
\boxed{\begin{array}{c}
\varkappa_1^e(a_j^e)\varkappa_2^e(a_j^e)\Big(-s_1\Big\{Q_0^e(a_j^e)\Big[\gimel_{-1}^1(a_j^e) \\
+\gimel_1^1(a_j^e)+\daleth_1^1(a_j^e)\daleth_{-1}^1(a_j^e)\Big]+Q_1^e(a_j^e) \\
\times\Big[\daleth_1^1(a_j^e)+\daleth_{-1}^1(a_j^e)\Big]+\tfrac{1}{2}Q_2^e(a_j^e) \\
+(Q_0^e(a_j^e))^{-1}\aleph_1^1(a_j^e)\aleph_{-1}^1(a_j^e)\Big\} \\
-t_1\Big\{Q_0^e(a_j^e)\Big[\aleph_{-1}^1(a_j^e)\beth_1^1(a_j^e) \\
+\aleph_1^1(a_j^e)\beth_{-1}^1(a_j^e)\Big]+(Q_0^e(a_j^e))^{-1} \\
\times\Big[\daleth_{-1}^1(a_j^e)+\daleth_1^1(a_j^e)-Q_1^e(a_j^e)(Q_0^e(a_j^e))^{-1}\Big] \\
+Q_1^e(a_j^e)\aleph_1^1(a_j^e)\aleph_{-1}^1(a_j^e)\Big\} \\
+\mathrm{i}(s_1+t_1)\Big\{\beth_{-1}^1(a_j^e)+\aleph_{-1}^1(a_j^e)\daleth_1^1(a_j^e) \\
-\beth_1^1(a_j^e)-\aleph_1^1(a_j^e)\daleth_{-1}^1(a_j^e)\Big\}\Big)
\end{array}} &
\boxed{\begin{array}{c}
(\varkappa_1^e(a_j^e))^2\Big(\mathrm{i}s_1\Big\{Q_0^e(a_j^e)\Big[2\gimel_1^1(a_j^e) \\
+(\daleth_1^1(a_j^e))^2\Big]+2Q_1^e(a_j^e)\daleth_1^1(a_j^e) \\
-(Q_0^e(a_j^e))^{-1}(\aleph_1^1(a_j^e))^2+\tfrac{1}{2}Q_2^e(a_j^e)\Big\} \\
+\mathrm{i}t_1\Big\{2Q_0^e(a_j^e)\aleph_1^1(a_j^e)\beth_1^1(a_j^e) \\
+Q_1^e(a_j^e)(\aleph_1^1(a_j^e))^2+(Q_0^e(a_j^e))^{-1} \\
\times\Big[Q_1^e(a_j^e)(Q_0^e(a_j^e))^{-1}-2\daleth_1^1(a_j^e)\Big]\Big\} \\
-2(s_1-t_1)\Big\{\beth_1^1(a_j^e)+\aleph_1^1(a_j^e)\daleth_1^1(a_j^e)\Big\}\Big)
\end{array}} \\
\boxed{\begin{array}{c}
(\varkappa_2^e(a_j^e))^2\Big(\mathrm{i}s_1\Big\{Q_0^e(a_j^e)\Big[2\gimel_{-1}^1(a_j^e) \\
+(\daleth_{-1}^1(a_j^e))^2\Big]+2Q_1^e(a_j^e)\daleth_{-1}^1(a_j^e) \\
-(Q_0^e(a_j^e))^{-1}(\aleph_{-1}^1(a_j^e))^2+\tfrac{1}{2}Q_2^e(a_j^e)\Big\} \\
+\mathrm{i}t_1\Big\{2Q_0^e(a_j^e)\aleph_{-1}^1(a_j^e)\beth_{-1}^1(a_j^e) \\
+Q_1^e(a_j^e)(\aleph_{-1}^1(a_j^e))^2+(Q_0^e(a_j^e))^{-1} \\
\times\Big[Q_1^e(a_j^e)(Q_0^e(a_j^e))^{-1}-2\daleth_{-1}^1(a_j^e)\Big]\Big\} \\
+2(s_1-t_1)\Big\{\beth_{-1}^1(a_j^e)+\aleph_{-1}^1(a_j^e)\daleth_{-1}^1(a_j^e)\Big\}\Big)
\end{array}} &
\boxed{\begin{array}{c}
\varkappa_1^e(a_j^e)\varkappa_2^e(a_j^e)\Big(s_1\Big\{Q_0^e(a_j^e)\Big[\gimel_{-1}^1(a_j^e) \\
+\gimel_1^1(a_j^e)+\daleth_1^1(a_j^e)\daleth_{-1}^1(a_j^e)\Big]+Q_1^e(a_j^e) \\
\times\Big[\daleth_1^1(a_j^e)+\daleth_{-1}^1(a_j^e)\Big]+\tfrac{1}{2}Q_2^e(a_j^e) \\
+(Q_0^e(a_j^e))^{-1}\aleph_1^1(a_j^e)\aleph_{-1}^1(a_j^e)\Big\} \\
+t_1\Big\{Q_0^e(a_j^e)\Big[\aleph_{-1}^1(a_j^e)\beth_1^1(a_j^e) \\
+\aleph_1^1(a_j^e)\beth_{-1}^1(a_j^e)\Big]+(Q_0^e(a_j^e))^{-1} \\
\times\Big[\daleth_{-1}^1(a_j^e)+\daleth_1^1(a_j^e)-Q_1^e(a_j^e)(Q_0^e(a_j^e))^{-1}\Big] \\
+Q_1^e(a_j^e)\aleph_1^1(a_j^e)\aleph_{-1}^1(a_j^e)\Big\} \\
+\mathrm{i}(s_1+t_1)\Big\{\beth_1^1(a_j^e)+\aleph_1^1(a_j^e)\daleth_{-1}^1(a_j^e) \\
-\beth_{-1}^1(a_j^e)-\aleph_{-1}^1(a_j^e)\daleth_1^1(a_j^e)\Big\}\Big)
\end{array}}
\end{pmatrix},$$

(with $\operatorname{tr}(\mathcal{C}^e(a_j^e))=0$), and $(f_k^e(n))_{ij} =_{n\to\infty} \mathcal{O}(1)$, $k\in\mathbb{N}$, $i,j=1,2$. (The expression for $\mathcal{C}^e(a_j^e)$ is necessary for obtaining asymptotics **at** the end-points $\{a_j^e\}_{j=1}^N$, as well as for Remark 5.2 below.) Returning to the counter-clockwise-oriented integrals $\oint_{\partial\mathbb{U}^e_{\delta_{a_j}}} \frac{w_+^{\Sigma_\circlearrowright^e}(s)}{s-z}\,\frac{\mathrm{d}s}{2\pi\mathrm{i}}$, $z\in\mathbb{C}\setminus\widetilde{\Sigma}_p^e$, it follows, via the Residue and Cauchy Theorems, that, for $j=1,\dots,N$,

$$\oint_{\partial\mathbb{U}^e_{\delta_{a_j}}} \frac{w_+^{\Sigma_\circlearrowright^e}(s)}{s-z}\,\frac{\mathrm{d}s}{2\pi\mathrm{i}} \underset{n\to\infty}{=} \begin{cases} -\dfrac{\widehat{\mathcal{A}^e}(a_j^e)}{n(z-a_j^e)^2}-\dfrac{\widehat{\mathcal{B}^e}(a_j^e)}{n(z-a_j^e)}+\mathcal{O}\left(\dfrac{\widehat{f^e}(z;n)}{n^2}\right), & z\in\mathbb{C}\setminus(\mathbb{U}^e_{\delta_{a_j}}\cup\partial\mathbb{U}^e_{\delta_{a_j}}), \\ -\dfrac{\widehat{\mathcal{A}^e}(a_j^e)}{n(z-a_j^e)^2}-\dfrac{\widehat{\mathcal{B}^e}(a_j^e)}{n(z-a_j^e)}+\dfrac{\mathcal{R}^{\infty}_{a_j^e}(z)}{n}+\mathcal{O}\left(\dfrac{\widehat{f^e}(z;n)}{n^2}\right), & z\in\mathbb{U}^e_{\delta_{a_j}}, \end{cases}$$

where $\widehat{\mathcal{A}^e}(a_j^e) := (\widehat{\alpha}_0^e(a_j^e))^{-1}\mathcal{A}^e(a_j^e)$, $\widehat{\mathcal{B}^e}(a_j^e) := (\widehat{\alpha}_0^e(a_j^e))^{-2}(\mathcal{B}^e(a_j^e)\widehat{\alpha}_0^e(a_j^e)-\mathcal{A}^e(a_j^e)\widehat{\alpha}_1^e(a_j^e))$, $\mathcal{R}^{\infty}_{a_j^e}(z)$ is given in Theorem 2.3.1, Equations (2.73) and (2.74), and $\widehat{f^e}(z;n)$, where the n-dependence arises due to the n-dependence of the associated Riemann theta functions, denotes some bounded (with respect to both z and n), analytic (for $\mathbb{C}\setminus\widetilde{\Sigma}_p^e\ni z$), $\mathrm{GL}_2(\mathbb{C})$-valued function for which $(\widehat{f^e}(z;n))_{kl} =_{\substack{n\to\infty \\ z\in\mathbb{C}\setminus\widetilde{\Sigma}_p^e}} \mathcal{O}(1)$, $k,l=1,2$. Repeating the above analysis for the remaining end-points of the support of the 'even' equilibrium measure, that is, $\{b_0^e,\dots,b_N^e,a_{N+1}^e\}$, one arrives at the result stated in the Lemma. □

Remark 5.2. A brisk perusing of the asymptotic (as $n\to\infty$) result for $\mathcal{R}^e(z)$ stated in Lemma 5.3 seems to imply that, at first glance, there are second-order poles at $\{b_{j-1}^e,a_j^e\}_{j=1}^{N+1}$; however, this is not the case.

As the proof of Lemma 5.3 demonstrates (cf. the analysis leading up to Equations (5.3)), Laurent series expansions about $\{b_{j-1}^{e},a_{j}^{e}\}_{j=1}^{N+1}$ show that, as $n \to \infty$, all expansions are, indeed, analytic; in particular: (i) for $z \in \mathbb{U}_{\delta_{a_j}}^{e}$, $j=1,\dots,N+1$ (all contour integrals are counter-clockwise oriented),

$$\oint_{\partial \mathbb{U}_{\delta_{a_j}}^{e}} \frac{w_{+}^{\Sigma_{\circlearrowright}^{e}}(s)}{s-z} \frac{\mathrm{d}s}{2\pi\mathrm{i}} \underset{\substack{n\to\infty \\ z\in\mathbb{U}_{\delta_{a_j}}^{e}}}{=} \frac{\left(\mathcal{A}^{e}(a_{j}^{e})\widehat{\alpha}_{0}^{e}(a_{j}^{e})\left(\left(\frac{\widehat{\alpha}_{1}^{e}(a_{j}^{e})}{\widehat{\alpha}_{0}^{e}(a_{j}^{e})}\right)^{2}-\frac{\widehat{\alpha}_{2}^{e}(a_{j}^{e})}{\widehat{\alpha}_{0}^{e}(a_{j}^{e})}\right)-\mathcal{B}^{e}(a_{j}^{e})\widehat{\alpha}_{1}^{e}(a_{j}^{e})+\mathcal{C}^{e}(a_{j}^{e})\widehat{\alpha}_{0}^{e}(a_{j}^{e})\right)}{n(\widehat{\alpha}_{0}^{e}(a_{j}^{e}))^{2}} + \frac{1}{n}\sum_{k=1}^{\infty} f_{k}^{a_{j}^{e}}(n)(z-a_{j}^{e})^{k} + \mathcal{O}\left(\frac{\widehat{f^{e}}(z;n)}{n^{2}}\right),$$

where, for $j=1,\dots,N$, $\mathcal{A}^{e}(a_{j}^{e})$, $\mathcal{B}^{e}(a_{j}^{e})$, $\mathcal{C}^{e}(a_{j}^{e})$, $\widehat{\alpha}_{0}^{e}(a_{j}^{e})$, $\widehat{\alpha}_{1}^{e}(a_{j}^{e})$, and $\widehat{\alpha}_{2}^{e}(a_{j}^{e})$ are given in (the proof of) Lemma 5.3, $\mathcal{A}^{e}(a_{N+1}^{e})$, $\mathcal{B}^{e}(a_{N+1}^{e})$ are given in Theorem 2.3.1, Equations (2.25), (2.27), (2.28), (2.31), (2.32), (2.37)–(2.45), (2.47), (2.52), and (2.53), $\widehat{\alpha}_{0}^{e}(a_{N+1}^{e}) := \frac{4}{3}f(a_{N+1}^{e})$, $\widehat{\alpha}_{1}^{e}(a_{N+1}^{e}) := \frac{4}{5}f'(a_{N+1}^{e})$, and $\widehat{\alpha}_{2}^{e}(a_{N+1}^{e}) := \frac{2}{7}f''(a_{N+1}^{e})$, with $f(a_{N+1}^{e})$, $f'(a_{N+1}^{e})$, and $f''(a_{N+1}^{e})$ given in Lemma 4.7, $\mathcal{C}^{e}(a_{N+1}^{e})$ is given by the same expression as $\mathcal{C}^{e}(a_{j}^{e})$ above subject to the modifications $\Omega_{j}^{e} \to 0$, $a_{j}^{e} \to a_{N+1}^{e}$, $Q_{0}^{e}(a_{j}^{e}) \to Q_{0}^{e}(a_{N+1}^{e})$, $Q_{1}^{e}(a_{j}^{e}) \to Q_{1}^{e}(a_{N+1}^{e})$, with $Q_{0}^{e}(a_{N+1}^{e})$, $Q_{1}^{e}(a_{N+1}^{e})$ given in Theorem 2.3.1, Equations (2.31) and (2.32), and $Q_{2}^{e}(a_{j}^{e}) \to Q_{2}^{e}(a_{N+1}^{e})$, where

$$Q_{2}^{e}(a_{N+1}^{e}) = -\frac{1}{2}Q_{0}^{e}(a_{N+1}^{e})\left(\sum_{k=1}^{N}\left(\frac{1}{(a_{N+1}^{e}-b_{k}^{e})^{2}}-\frac{1}{(a_{N+1}^{e}-a_{k}^{e})^{2}}\right)+\frac{1}{(a_{N+1}^{e}-b_{0}^{e})^{2}}\right) + \frac{1}{4}Q_{0}^{e}(a_{N+1}^{e})\left(\sum_{k=1}^{N}\left(\frac{1}{a_{N+1}^{e}-b_{k}^{e}}-\frac{1}{a_{N+1}^{e}-a_{k}^{e}}\right)+\frac{1}{a_{N+1}^{e}-b_{0}^{e}}\right)^{2},$$

$\widehat{f^{e}}(z;n)$ is characterised completely at the end of the proof of Lemma 5.3, and $(f_{k}^{a_{j}^{e}}(n))_{l_1 l_2} =_{n\to\infty} \mathcal{O}(1)$, $k\in\mathbb{N}$, $l_1,l_2=1,2$; and (ii) for $z\in\mathbb{U}_{\delta_{b_j}}^{e}$, $j=0,\dots,N$,

$$\oint_{\partial \mathbb{U}_{\delta_{b_j}}^{e}} \frac{w_{+}^{\Sigma_{\circlearrowright}^{e}}(s)}{s-z} \frac{\mathrm{d}s}{2\pi\mathrm{i}} \underset{\substack{n\to\infty \\ z\in\mathbb{U}_{\delta_{b_j}}^{e}}}{=} \frac{\left(\mathcal{A}^{e}(b_{j}^{e})\widehat{\alpha}_{0}^{e}(b_{j}^{e})\left(\left(\frac{\widehat{\alpha}_{1}^{e}(b_{j}^{e})}{\widehat{\alpha}_{0}^{e}(b_{j}^{e})}\right)^{2}-\frac{\widehat{\alpha}_{2}^{e}(b_{j}^{e})}{\widehat{\alpha}_{0}^{e}(b_{j}^{e})}\right)-\mathcal{B}^{e}(b_{j}^{e})\widehat{\alpha}_{1}^{e}(b_{j}^{e})+\mathcal{C}^{e}(b_{j}^{e})\widehat{\alpha}_{0}^{e}(b_{j}^{e})\right)}{n(\widehat{\alpha}_{0}^{e}(b_{j}^{e}))^{2}} + \frac{1}{n}\sum_{k=1}^{\infty} f_{k}^{b_{j}^{e}}(n)(z-b_{j}^{e})^{k} + \mathcal{O}\left(\frac{\widehat{f^{e}}(z;n)}{n^{2}}\right),$$

where, for $j=1,\dots,N+1$, $\mathcal{A}^{e}(b_{j-1}^{e})$, $\mathcal{B}^{e}(b_{j-1}^{e})$ are given in Theorem 2.3.1, Equations (2.24), (2.26), (2.28), (2.29), (2.30), (2.33), (2.34), (2.37)–(2.45), (2.46), (2.48), (2.50), (2.51), (2.54), and (2.55), $\widehat{\alpha}_{0}^{e}(b_{j-1}^{e}) := \frac{4}{3}f(b_{j-1}^{e})$, $\widehat{\alpha}_{1}^{e}(b_{j-1}^{e}) := \frac{4}{5}f'(b_{j-1}^{e})$, and $\widehat{\alpha}_{2}^{e}(b_{j-1}^{e}) := \frac{2}{7}f''(b_{j-1}^{e})$, with $f(b_{j-1}^{e})$, $f'(b_{j-1}^{e})$, and $f''(b_{j-1}^{e})$ given in Lemma 4.6,

for $j=1,\dots,N$,

$$\frac{\mathcal{C}^e(b_j^e)}{\mathrm{e}^{\mathrm{i}n\Omega_j^e}} := \begin{pmatrix} \boxed{\begin{array}{c} \varkappa_1^e(b_j^e)\varkappa_2^e(b_j^e)\Big(s_1\Big\{-Q_0^e(b_j^e)\aleph_1^1(b_j^e)\aleph_{-1}^1(b_j^e) \\ -(Q_0^e(b_j^e))^{-3}(Q_1^e(b_j^e))^2+\frac{1}{2}Q_2^e(b_j^e)(Q_0^e(b_j^e))^{-2} \\ +Q_1^e(b_j^e)(Q_0^e(b_j^e))^{-2}\Big[\daleth_1^1(b_j^e)+\daleth_{-1}^1(b_j^e)\Big] \\ -(Q_0^e(b_j^e))^{-1}\Big[\gimel_1^1(b_j^e)+\gimel_{-1}^1(b_j^e)+\daleth_1^1(b_j^e)\daleth_{-1}^1(b_j^e)\Big]\Big\} \\ +t_1\Big\{-Q_1^e(b_j^e)-Q_0^e(b_j^e)\Big[\daleth_{-1}^1(b_j^e)+\daleth_1^1(b_j^e)\Big] \\ +Q_1^e(b_j^e)(Q_0^e(b_j^e))^{-2}\aleph_1^1(b_j^e)\aleph_{-1}^1(b_j^e)-(Q_0^e(b_j^e))^{-1} \\ \times\Big[\aleph_1^1(b_j^e)\beth_{-1}^1(b_j^e)+\aleph_{-1}^1(b_j^e)\beth_1^1(b_j^e)\Big]\Big\} \\ +\mathrm{i}(s_1+t_1)\Big[\beth_{-1}^1(b_j^e)-\aleph_1^1(b_j^e)\daleth_{-1}^1(b_j^e) \\ +\aleph_{-1}^1(b_j^e)\daleth_1^1(b_j^e)-\beth_1^1(b_j^e)\Big]\Big) \end{array}} & \boxed{\begin{array}{c} (\varkappa_1^e(b_j^e))^2\Big(\mathrm{i}s_1\Big\{Q_0^e(b_j^e)(\aleph_1^1(b_j^e))^2-(Q_0^e(b_j^e))^{-3} \\ \times(Q_1^e(b_j^e))^2+\frac{1}{2}Q_2^e(b_j^e)(Q_0^e(b_j^e))^{-2}+2Q_1^e(b_j^e) \\ \times(Q_0^e(b_j^e))^{-2}\daleth_1^1(b_j^e)-(Q_0^e(b_j^e))^{-1}\Big[2\gimel_1^1(b_j^e) \\ +(\daleth_1^1(b_j^e))^2\Big]\Big\}+\mathrm{i}t_1\Big\{2Q_0^e(b_j^e)\daleth_1^1(b_j^e)+Q_1^e(b_j^e) \\ +Q_1^e(b_j^e)(Q_0^e(b_j^e))^{-2}(\aleph_1^1(b_j^e))^2-2(Q_0^e(b_j^e))^{-1} \\ \times\aleph_1^1(b_j^e)\beth_1^1(b_j^e)\Big\}+2(s_1-t_1) \\ \times\Big\{\beth_1^1(b_j^e)+\aleph_1^1(b_j^e)\daleth_1^1(b_j^e)\Big\}\Big) \end{array}} \\ \boxed{\begin{array}{c} (\varkappa_2^e(b_j^e))^2\Big(\mathrm{i}s_1\Big\{Q_0^e(b_j^e)(\aleph_{-1}^1(b_j^e))^2-(Q_0^e(b_j^e))^{-3} \\ \times(Q_1^e(b_j^e))^2+\frac{1}{2}Q_2^e(b_j^e)(Q_0^e(b_j^e))^{-2}+2Q_1^e(b_j^e) \\ \times(Q_0^e(b_j^e))^{-2}\daleth_{-1}^1(b_j^e)-(Q_0^e(b_j^e))^{-1}\Big[2\gimel_{-1}^1(b_j^e) \\ +(\daleth_{-1}^1(b_j^e))^2\Big]\Big\}+\mathrm{i}t_1\Big\{2Q_0^e(b_j^e)\daleth_{-1}^1(b_j^e)+Q_1^e(b_j^e) \\ +Q_1^e(b_j^e)(Q_0^e(b_j^e))^{-2}(\aleph_{-1}^1(b_j^e))^2-2(Q_0^e(b_j^e))^{-1} \\ \times\aleph_{-1}^1(b_j^e)\beth_{-1}^1(b_j^e)\Big\}-2(s_1-t_1) \\ \times\Big\{\beth_{-1}^1(b_j^e)+\aleph_{-1}^1(b_j^e)\daleth_{-1}^1(b_j^e)\Big\}\Big) \end{array}} & \boxed{\begin{array}{c} \varkappa_1^e(b_j^e)\varkappa_2^e(b_j^e)\Big(s_1\Big\{Q_0^e(b_j^e)\aleph_1^1(b_j^e)\aleph_{-1}^1(b_j^e) \\ +(Q_0^e(b_j^e))^{-3}(Q_1^e(b_j^e))^2-\frac{1}{2}Q_2^e(b_j^e)(Q_0^e(b_j^e))^{-2} \\ -Q_1^e(b_j^e)(Q_0^e(b_j^e))^{-2}\Big[\daleth_1^1(b_j^e)+\daleth_{-1}^1(b_j^e)\Big] \\ +(Q_0^e(b_j^e))^{-1}\Big[\gimel_1^1(b_j^e)+\gimel_{-1}^1(b_j^e)+\daleth_1^1(b_j^e)\daleth_{-1}^1(b_j^e)\Big]\Big\} \\ +t_1\Big\{Q_1^e(b_j^e)+Q_0^e(b_j^e)\Big[\daleth_{-1}^1(b_j^e)+\daleth_1^1(b_j^e)\Big] \\ -Q_1^e(b_j^e)(Q_0^e(b_j^e))^{-2}\aleph_1^1(b_j^e)\aleph_{-1}^1(b_j^e)+(Q_0^e(b_j^e))^{-1} \\ \times\Big[\aleph_1^1(b_j^e)\beth_{-1}^1(b_j^e)+\aleph_{-1}^1(b_j^e)\beth_1^1(b_j^e)\Big]\Big\} \\ +\mathrm{i}(s_1+t_1)\Big[\beth_1^1(b_j^e)-\aleph_{-1}^1(b_j^e)\daleth_1^1(b_j^e) \\ +\aleph_1^1(b_j^e)\daleth_{-1}^1(b_j^e)-\beth_{-1}^1(b_j^e)\Big]\Big) \end{array}} \end{pmatrix}$$

(with $\operatorname{tr}(\mathcal{C}^e(b_j^e))=0$), where $Q_0^e(b_j^e)$, $Q_1^e(b_j^e)$ are given in Theorem 2.3.1, Equations (2.33) and (2.34),

$$\begin{aligned} Q_2^e(b_j^e) &= -\frac{1}{2}Q_0^e(b_j^e)\left(\sum_{\substack{k=1\\k\neq j}}^{N}\left(\frac{1}{(b_j^e-b_k^e)^2}-\frac{1}{(b_j^e-a_k^e)^2}\right)+\frac{1}{(b_j^e-b_0^e)^2}-\frac{1}{(b_j^e-a_{N+1}^e)^2}-\frac{1}{(b_j^e-a_j^e)^2}\right) \\ &+ \frac{1}{4}Q_0^e(b_j^e)\left(\sum_{\substack{k=1\\k\neq j}}^{N}\left(\frac{1}{b_j^e-b_k^e}-\frac{1}{b_j^e-a_k^e}\right)+\frac{1}{b_j^e-b_0^e}-\frac{1}{b_j^e-a_{N+1}^e}-\frac{1}{b_j^e-a_j^e}\right)^2, \end{aligned}$$

$\mathcal{C}^e(b_0^e)$ is given by the same expression as $\mathcal{C}^e(b_j^e)$ above subject to the modifications $\Omega_j^e\to 0$, $b_j^e\to b_0^e$, $Q_0^e(b_j^e)\to Q_0^e(b_0^e)$, $Q_1^e(b_j^e)\to Q_1^e(b_0^e)$, with $Q_0^e(b_0^e)$, $Q_1^e(b_0^e)$ given in Theorem 2.3.1, Equations (2.29) and (2.30), and $Q_2^e(b_j^e)\to Q_2^e(b_0^e)$, where

$$\begin{aligned} Q_2^e(b_0^e) &= -\frac{1}{2}Q_0^e(b_0^e)\left(\sum_{k=1}^{N}\left(\frac{1}{(b_0^e-b_k^e)^2}-\frac{1}{(b_0^e-a_k^e)^2}\right)-\frac{1}{(b_0^e-a_{N+1}^e)^2}\right) \\ &+ \frac{1}{4}Q_0^e(b_0^e)\left(\sum_{k=1}^{N}\left(\frac{1}{b_0^e-b_k^e}-\frac{1}{b_0^e-a_k^e}\right)-\frac{1}{b_0^e-a_{N+1}^e}\right)^2, \end{aligned}$$

and $(f_k^{b_{j-1}^e}(n))_{l_1 l_2} =_{n\to\infty} \mathcal{O}(1)$, $j=1,\dots,N+1$, $k\in\mathbb{N}$, $l_1,l_2=1,2$. ∎

Re-tracing the finite sequence of RHP transformations (all of which are invertible) and definitions, namely, $\mathcal{R}^e(z)$ (Lemmas 5.3 and 4.8) and $\mathcal{S}_p^e(z)$ (Lemma 4.8) $\to$ $\mathcal{X}^e(z)$ (Lemmas 4.6 and 4.7) $\to$ $\overset{e}{m}{}^{\infty}(z)$ (Lemma 4.5) $\to$ $\overset{e}{\mathcal{M}}{}^{\sharp}(z)$ (Lemma 4.2) $\to$ $\overset{e}{\mathcal{M}}(z)$ (Lemma 4.1) $\to$ $\overset{e}{\mathrm{Y}}(z)$ (Lemma 3.4), the asymptotic (as $n\to\infty$) solution of the original **RHP1**, that is, $(\overset{e}{\mathrm{Y}}(z), \mathrm{I}+\exp(-n\widetilde{V}(z))\sigma_+, \mathbb{R})$, in the various bounded and unbounded regions (Figure 7), is given by:

(1) for $z \in \Upsilon_1^e \cup \Upsilon_2^e$,

$$\overset{e}{\mathrm{Y}}(z) = \mathrm{e}^{\frac{n\ell_e}{2}\,\mathrm{ad}(\sigma_3)} \mathcal{R}^e(z) \overset{e}{m}^{\infty}(z) \mathrm{e}^{n(g^e(z) + \int_{J_e} \ln(s) \psi_V^e(s)\,\mathrm{d}s)\sigma_3},$$

where $g^e(z)$, $\psi_V^e(z)$, ℓ_e, $\overset{e}{m}^{\infty}(z)$, and $\mathcal{R}^e(z)$ are given in Lemmas 3.4, 3.5, 3.6, 4.5, and 5.3, respectively;

(2) for $z \in \Upsilon_3^e$,

$$\overset{e}{\mathrm{Y}}(z) = \mathrm{e}^{\frac{n\ell_e}{2}\,\mathrm{ad}(\sigma_3)} \mathcal{R}^e(z) \overset{e}{m}^{\infty}(z) \Big(\mathrm{I} + \mathrm{e}^{-4n\pi\mathrm{i}\int_z^{a_{N+1}^e} \psi_V^e(s)\,\mathrm{d}s}\,\sigma_-\Big) \mathrm{e}^{n(g^e(z) + \int_{J_e} \ln(s) \psi_V^e(s)\,\mathrm{d}s)\sigma_3};$$

(3) for $z \in \Upsilon_4^e$,

$$\overset{e}{\mathrm{Y}}(z) = \mathrm{e}^{\frac{n\ell_e}{2}\,\mathrm{ad}(\sigma_3)} \mathcal{R}^e(z) \overset{e}{m}^{\infty}(z) \Big(\mathrm{I} - \mathrm{e}^{4n\pi\mathrm{i}\int_z^{a_{N+1}^e} \psi_V^e(s)\,\mathrm{d}s}\,\sigma_-\Big) \mathrm{e}^{n(g^e(z) + \int_{J_e} \ln(s) \psi_V^e(s)\,\mathrm{d}s)\sigma_3};$$

(4) for $z \in (\Omega_{b_{j-1}}^{e,1} \cup \Omega_{b_{j-1}}^{e,4}) \cup (\Omega_{a_j}^{e,1} \cup \Omega_{a_j}^{e,4})$, $j = 1, \ldots, N+1$,

$$\overset{e}{\mathrm{Y}}(z) = \mathrm{e}^{\frac{n\ell_e}{2}\,\mathrm{ad}(\sigma_3)} \mathcal{R}^e(z) \mathcal{X}^e(z) \mathrm{e}^{n(g^e(z) + \int_{J_e} \ln(s) \psi_V^e(s)\,\mathrm{d}s)\sigma_3},$$

where, for $z \in \mathbb{U}_{\delta_{b_{j-1}}}^e$ $(\supset \Omega_{b_{j-1}}^{e,1} \cup \Omega_{b_{j-1}}^{e,4})$, $\mathcal{X}^e(z)$ is given by Lemma 4.6, and, for $z \in \mathbb{U}_{\delta_{a_j}}^e$ $(\supset \Omega_{a_j}^{e,1} \cup \Omega_{a_j}^{e,4})$, $\mathcal{X}^e(z)$ is given by Lemma 4.7;

(5) for $z \in \Omega_{b_{j-1}}^{e,2} \cup \Omega_{a_j}^{e,2}$, $j = 1, \ldots, N+1$,

$$\overset{e}{\mathrm{Y}}(z) = \mathrm{e}^{\frac{n\ell_e}{2}\,\mathrm{ad}(\sigma_3)} \mathcal{R}^e(z) \mathcal{X}^e(z) \Big(\mathrm{I} + \mathrm{e}^{-4n\pi\mathrm{i}\int_z^{a_{N+1}^e} \psi_V^e(s)\,\mathrm{d}s}\,\sigma_-\Big) \mathrm{e}^{n(g^e(z) + \int_{J_e} \ln(s) \psi_V^e(s)\,\mathrm{d}s)\sigma_3};$$

(6) for $z \in \Omega_{b_{j-1}}^{e,3} \cup \Omega_{a_j}^{e,3}$, $j = 1, \ldots, N+1$,

$$\overset{e}{\mathrm{Y}}(z) = \mathrm{e}^{\frac{n\ell_e}{2}\,\mathrm{ad}(\sigma_3)} \mathcal{R}^e(z) \mathcal{X}^e(z) \Big(\mathrm{I} - \mathrm{e}^{4n\pi\mathrm{i}\int_z^{a_{N+1}^e} \psi_V^e(s)\,\mathrm{d}s}\,\sigma_-\Big) \mathrm{e}^{n(g^e(z) + \int_{J_e} \ln(s) \psi_V^e(s)\,\mathrm{d}s)\sigma_3}.$$

Multiplying the respective matrices in items (1)–(6) above and collecting (1 1)- and (1 2)-elements, one arrives at, finally, the asymptotic (as $n \to \infty$) results for $\boldsymbol{\pi}_{2n}(z)$ and $\int_{\mathbb{R}} \frac{\boldsymbol{\pi}_{2n}(s)\exp(-n\widetilde{V}(s))}{s-z} \frac{\mathrm{d}s}{2\pi\mathrm{i}}$ (in the entire complex plane) stated in Theorem 2.3.1.

In order to obtain asymptotics (as $n \to \infty$) for $\xi_n^{(2n)}$ $(= \|\boldsymbol{\pi}_{2n}(\cdot)\|_{\mathcal{L}}^{-1} = (H_{2n}^{(-2n)}/H_{2n+1}^{(-2n)})^{1/2})$ and $\phi_{2n}(z)$ $(= \xi_n^{(2n)} \boldsymbol{\pi}_{2n}(z))$ stated in Theorem 2.3.2, large-z asymptotics for $\overset{e}{\mathrm{Y}}(z)$ are necessary.

Proposition 5.3. *Let* $\mathcal{R}^e \colon \mathbb{C} \setminus \widetilde{\Sigma}_p^e \to \mathrm{SL}_2(\mathbb{C})$ *be the solution of the* RHP $(\mathcal{R}^e(z), v_{\mathcal{R}}^e(z), \widetilde{\Sigma}_p^e)$ *formulated in Proposition* 5.2 *with* $n \to \infty$ *asymptotics given in Lemma* 5.3. *Then,*

$$\mathcal{R}^e(z) \underset{z \to \infty}{=} \mathrm{I} + \frac{1}{z}\mathcal{R}_1^{e,\infty}(n) + \frac{1}{z^2}\mathcal{R}_2^{e,\infty}(n) + O\Big(\frac{1}{z^3}\Big),$$

where, for $k = 1, 2$,

$$\mathcal{R}_k^{e,\infty}(n) := -\int_{\Sigma_{\circlearrowright}^e} s^{k-1} w_+^{\Sigma_{\circlearrowright}^e}(s) \frac{\mathrm{d}s}{2\pi\mathrm{i}} = \sum_{j=1}^{N+1} \sum_{q \in \{b_{j-1}^e, a_j^e\}} \mathrm{Res}\Big(z^{k-1} w_+^{\Sigma_{\circlearrowright}^e}(z); q\Big),$$

with, in particular,

$$\begin{aligned}
\mathcal{R}_1^{e,\infty}(n) \underset{n \to \infty}{=} \frac{1}{n} \sum_{j=1}^{N+1} \Bigg(&\frac{(\mathcal{B}^e(a_j^e)\widehat{\alpha}_0^e(a_j^e) - \mathcal{A}^e(a_j^e)\widehat{\alpha}_1^e(a_j^e))}{(\widehat{\alpha}_0^e(a_j^e))^2} + \frac{(\mathcal{B}^e(b_{j-1}^e)\widehat{\alpha}_0^e(b_{j-1}^e) - \mathcal{A}^e(b_{j-1}^e)\widehat{\alpha}_1^e(b_{j-1}^e))}{(\widehat{\alpha}_0^e(b_{j-1}^e))^2} \Bigg) \\
&+ O\Big(\frac{1}{n^2}\Big), \\
\mathcal{R}_2^{e,\infty}(n) \underset{n \to \infty}{=} \frac{1}{n} \sum_{j=1}^{N+1} \Bigg(&\frac{(\widehat{\alpha}_0^e(b_{j-1}^e)\mathcal{A}^e(b_{j-1}^e) + b_{j-1}^e(\mathcal{B}^e(b_{j-1}^e)\widehat{\alpha}_0^e(b_{j-1}^e) - \mathcal{A}^e(b_{j-1}^e)\widehat{\alpha}_1^e(b_{j-1}^e)))}{(\widehat{\alpha}_0^e(b_{j-1}^e))^2}
\end{aligned}$$

$$+\left.\frac{(\widehat{\alpha}_0^e(a_j^e)\mathcal{A}^e(a_j^e)+a_j^e(\mathcal{B}^e(a_j^e)\widehat{\alpha}_0^e(a_j^e)-\mathcal{A}^e(a_j^e)\widehat{\alpha}_1^e(a_j^e)))}{(\widehat{\alpha}_0^e(a_j^e))^2}\right)+\mathcal{O}\left(\frac{1}{n^2}\right),$$

and all the parameters defined in Lemma 5.3.

Let $\overset{e}{m}^{\infty}\colon \mathbb{C}\setminus J_e^{\infty}\to \mathrm{SL}_2(\mathbb{C})$ *solve the* RHP $(\overset{e}{m}^{\infty}(z),J_e^{\infty},\overset{e}{v}^{\infty}(z))$ *formulated in Lemma* 4.3 *with (unique) solution given by Lemma* 4.5. *For* $\varepsilon_1,\varepsilon_2=\pm1$, *set*

$$\theta_{\infty}^e(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^e):=\boldsymbol{\theta}^e(\varepsilon_1\boldsymbol{u}_+^e(\infty)-\tfrac{n}{2\pi}\boldsymbol{\Omega}^e+\varepsilon_2\boldsymbol{d}_e),$$
$$\alpha_{\infty}^e(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^e):=2\pi\mathrm{i}\varepsilon_1\sum_{m\in\mathbb{Z}^N}(m,\widehat{\boldsymbol{\alpha}}_{\infty}^e)\mathrm{e}^{2\pi\mathrm{i}(m,\varepsilon_1\boldsymbol{u}_+^e(\infty)-\frac{n}{2\pi}\boldsymbol{\Omega}^e+\varepsilon_2\boldsymbol{d}_e)+\pi\mathrm{i}(m,\tau^e m)},$$

where $\widehat{\boldsymbol{\alpha}}_{\infty}^e=(\widehat{\alpha}_{\infty,1}^e,\widehat{\alpha}_{\infty,2}^e,\dots,\widehat{\alpha}_{\infty,N}^e)$, *with* $\widehat{\alpha}_{\infty,j}^e:=c_{j1}^e$, $j=1,\dots,N$, *and*

$$\beta_{\infty}^e(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^e):=2\pi\sum_{m\in\mathbb{Z}^N}\left(\pi(m,\widehat{\boldsymbol{\alpha}}_{\infty}^e)^2+\mathrm{i}\varepsilon_1(m,\widehat{\boldsymbol{\beta}}_{\infty}^e)\right)\mathrm{e}^{2\pi\mathrm{i}(m,\varepsilon_1\boldsymbol{u}_+^e(\infty)-\frac{n}{2\pi}\boldsymbol{\Omega}^e+\varepsilon_2\boldsymbol{d}_e)+\pi\mathrm{i}(m,\tau^e m)},$$

where $\widehat{\boldsymbol{\beta}}_{\infty}^e=(\widehat{\beta}_{\infty,1}^e,\widehat{\beta}_{\infty,2}^e,\dots,\widehat{\beta}_{\infty,N}^e)$, *with* $\widehat{\beta}_{\infty,j}^e:=\frac{1}{2}(c_{j2}^e+\frac{1}{2}c_{j1}^e\sum_{k=1}^{N+1}(b_{k-1}^e+a_k^e))$, $j=1,\dots,N$, *where* c_{j1}^e,c_{j2}^e, $j=1,\dots,N$, *are obtained from Equations* (E1) *and* (E2). *Then,*

$$\overset{e}{m}^{\infty}(z)\underset{z\to\infty}{=}\mathrm{I}+\frac{1}{z}\overset{e}{m}_1^{\infty}+\frac{1}{z^2}\overset{e}{m}_2^{\infty}+\mathcal{O}\left(\frac{1}{z^3}\right),$$

where

$$(\overset{e}{m}_1^{\infty})_{11}=\frac{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\infty)+\boldsymbol{d}_e)}{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\infty)-\frac{n}{2\pi}\boldsymbol{\Omega}^e+\boldsymbol{d}_e)}\left(\frac{\theta_{\infty}^e(1,1,\boldsymbol{\Omega}^e)\alpha_{\infty}^e(1,1,\vec{\boldsymbol{0}})-\alpha_{\infty}^e(1,1,\boldsymbol{\Omega}^e)\theta_{\infty}^e(1,1,\vec{\boldsymbol{0}})}{(\theta_{\infty}^e(1,1,\vec{\boldsymbol{0}}))^2}\right),$$

$$(\overset{e}{m}_1^{\infty})_{12}=\frac{1}{4\mathrm{i}}\left(\sum_{k=1}^{N+1}(b_{k-1}^e-a_k^e)\right)\frac{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\infty)+\boldsymbol{d}_e)\theta_{\infty}^e(-1,1,\boldsymbol{\Omega}^e)}{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\infty)-\frac{n}{2\pi}\boldsymbol{\Omega}^e+\boldsymbol{d}_e)\theta_{\infty}^e(-1,1,\vec{\boldsymbol{0}})},$$

$$(\overset{e}{m}_1^{\infty})_{21}=-\frac{1}{4\mathrm{i}}\left(\sum_{k=1}^{N+1}(b_{k-1}^e-a_k^e)\right)\frac{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\infty)+\boldsymbol{d}_e)\theta_{\infty}^e(1,-1,\boldsymbol{\Omega}^e)}{\boldsymbol{\theta}^e(-\boldsymbol{u}_+^e(\infty)-\frac{n}{2\pi}\boldsymbol{\Omega}^e-\boldsymbol{d}_e)\theta_{\infty}^e(1,-1,\vec{\boldsymbol{0}})},$$

$$\begin{aligned}(\overset{e}{m}_1^{\infty})_{22}=&\left(\frac{\theta_{\infty}^e(-1,-1,\boldsymbol{\Omega}^e)\alpha_{\infty}^e(-1,-1,\vec{\boldsymbol{0}})-\alpha_{\infty}^e(-1,-1,\boldsymbol{\Omega}^e)\theta_{\infty}^e(-1,-1,\vec{\boldsymbol{0}})}{(\theta_{\infty}^e(-1,-1,\vec{\boldsymbol{0}}))^2}\right)\\&\times\frac{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\infty)+\boldsymbol{d}_e)}{\boldsymbol{\theta}^e(-\boldsymbol{u}_+^e(\infty)-\frac{n}{2\pi}\boldsymbol{\Omega}^e-\boldsymbol{d}_e)},\end{aligned}$$

$$\begin{aligned}(\overset{e}{m}_2^{\infty})_{11}=&\Big(\theta_{\infty}^e(1,1,\boldsymbol{\Omega}^e)\Big(\beta_{\infty}^e(1,1,\vec{\boldsymbol{0}})\theta_{\infty}^e(1,1,\vec{\boldsymbol{0}})+(\alpha_{\infty}^e(1,1,\vec{\boldsymbol{0}}))^2\Big)-\alpha_{\infty}^e(1,1,\boldsymbol{\Omega}^e)\alpha_{\infty}^e(1,1,\vec{\boldsymbol{0}})\theta_{\infty}^e(1,1,\vec{\boldsymbol{0}})\\&-\beta_{\infty}^e(1,1,\boldsymbol{\Omega}^e)(\theta_{\infty}^e(1,1,\vec{\boldsymbol{0}}))^2\Big)\frac{(\theta_{\infty}^e(1,1,\vec{\boldsymbol{0}}))^{-3}\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\infty)+\boldsymbol{d}_e)}{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\infty)-\frac{n}{2\pi}\boldsymbol{\Omega}^e+\boldsymbol{d}_e)}+\frac{1}{32}\left(\sum_{k=1}^{N+1}(b_{k-1}^e-a_k^e)\right)^2,\end{aligned}$$

$$\begin{aligned}(\overset{e}{m}_2^{\infty})_{12}=&\frac{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\infty)+\boldsymbol{d}_e)}{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\infty)-\frac{n}{2\pi}\boldsymbol{\Omega}^e+\boldsymbol{d}_e)}\left(\left(\frac{\theta_{\infty}^e(-1,1,\boldsymbol{\Omega}^e)\alpha_{\infty}^e(-1,1,\vec{\boldsymbol{0}})-\alpha_{\infty}^e(-1,1,\boldsymbol{\Omega}^e)\theta_{\infty}^e(-1,1,\vec{\boldsymbol{0}})}{(\theta_{\infty}^e(-1,1,\vec{\boldsymbol{0}}))^2}\right)\right.\\&\left.\times\frac{1}{4\mathrm{i}}\left(\sum_{k=1}^{N+1}(b_{k-1}^e-a_k^e)\right)+\frac{1}{8\mathrm{i}}\left(\sum_{k=1}^{N+1}((b_{k-1}^e)^2-(a_k^e)^2)\right)\frac{\theta_{\infty}^e(-1,1,\boldsymbol{\Omega}^e)}{\theta_{\infty}^e(-1,1,\vec{\boldsymbol{0}})}\right),\end{aligned}$$

$$\begin{aligned}(\overset{e}{m}_2^{\infty})_{21}=&-\frac{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\infty)+\boldsymbol{d}_e)}{\boldsymbol{\theta}^e(-\boldsymbol{u}_+^e(\infty)-\frac{n}{2\pi}\boldsymbol{\Omega}^e-\boldsymbol{d}_e)}\left(\left(\frac{\theta_{\infty}^e(1,-1,\boldsymbol{\Omega}^e)\alpha_{\infty}^e(1,-1,\vec{\boldsymbol{0}})-\alpha_{\infty}^e(1,-1,\boldsymbol{\Omega}^e)\theta_{\infty}^e(1,-1,\vec{\boldsymbol{0}})}{(\theta_{\infty}^e(1,-1,\vec{\boldsymbol{0}}))^2}\right)\right.\\&\left.\times\frac{1}{4\mathrm{i}}\left(\sum_{k=1}^{N+1}(b_{k-1}^e-a_k^e)\right)+\frac{1}{8\mathrm{i}}\left(\sum_{k=1}^{N+1}((b_{k-1}^e)^2-(a_k^e)^2)\right)\frac{\theta_{\infty}^e(1,-1,\boldsymbol{\Omega}^e)}{\theta_{\infty}^e(1,-1,\vec{\boldsymbol{0}})}\right),\end{aligned}$$

$$\begin{aligned}(\overset{e}{m}_2^{\infty})_{22}=&\Big(\theta_{\infty}^e(-1,-1,\boldsymbol{\Omega}^e)\Big(\beta_{\infty}^e(-1,-1,\vec{\boldsymbol{0}})\theta_{\infty}^e(-1,-1,\vec{\boldsymbol{0}})+(\alpha_{\infty}^e(-1,-1,\vec{\boldsymbol{0}}))^2\Big)-\alpha_{\infty}^e(-1,-1,\boldsymbol{\Omega}^e)\\&\times\alpha_{\infty}^e(-1,-1,\vec{\boldsymbol{0}})\theta_{\infty}^e(-1,-1,\vec{\boldsymbol{0}})-\beta_{\infty}^e(-1,-1,\boldsymbol{\Omega}^e)(\theta_{\infty}^e(-1,-1,\vec{\boldsymbol{0}}))^2\Big)(\theta_{\infty}^e(-1,-1,\vec{\boldsymbol{0}}))^{-3}\end{aligned}$$

$$\times \frac{\boldsymbol{\theta}^{e}(\boldsymbol{u}^{e}_{+}(\infty)+\boldsymbol{d}_{e})}{\boldsymbol{\theta}^{e}(-\boldsymbol{u}^{e}_{+}(\infty)-\frac{n}{2\pi}\boldsymbol{\Omega}^{e}-\boldsymbol{d}_{e})}+\frac{1}{32}\left(\sum_{k=1}^{N+1}(b^{e}_{k-1}-a^{e}_{k})\right)^{2},$$

with $(\star)_{ij}$, $i,j=1,2$, *denoting the* $(i\,j)$*-element of* $\star$, *and* $\vec{\boldsymbol{0}}:=(0,0,\dots,0)^{\mathrm{T}}$ $(\in\mathbb{R}^{N})$.

Let $\overset{e}{\mathrm{Y}}\colon \mathbb{C}\setminus\mathbb{R}\to\mathrm{SL}_{2}(\mathbb{C})$ *be the solution of* **RHP1**. *Then,*

$$\overset{e}{\mathrm{Y}}(z)z^{-n\sigma_{3}} \underset{z\to\infty}{=} \mathrm{I}+\frac{1}{z}\mathrm{Y}^{e,\infty}_{1}+\frac{1}{z^{2}}\mathrm{Y}^{e,\infty}_{2}+\mathcal{O}\left(\frac{1}{z^{3}}\right),$$

where

$$\begin{aligned}
(\mathrm{Y}^{e,\infty}_{1})_{11} &= -2n\int_{J_{e}}s\psi^{e}_{V}(s)\,\mathrm{d}s+(\overset{e}{m}{}^{\infty}_{1})_{11}+(\mathcal{R}^{e,\infty}_{1}(n))_{11},\\
(\mathrm{Y}^{e,\infty}_{1})_{12} &= \mathrm{e}^{n\ell_{e}}\Big((\overset{e}{m}{}^{\infty}_{1})_{12}+(\mathcal{R}^{e,\infty}_{1}(n))_{12}\Big),\\
(\mathrm{Y}^{e,\infty}_{1})_{21} &= \mathrm{e}^{-n\ell_{e}}\Big((\overset{e}{m}{}^{\infty}_{1})_{21}+(\mathcal{R}^{e,\infty}_{1}(n))_{21}\Big),\\
(\mathrm{Y}^{e,\infty}_{1})_{22} &= 2n\int_{J_{e}}s\psi^{e}_{V}(s)\,\mathrm{d}s+(\overset{e}{m}{}^{\infty}_{1})_{22}+(\mathcal{R}^{e,\infty}_{1}(n))_{22},\\
(\mathrm{Y}^{e,\infty}_{2})_{11} &= 2n^{2}\left(\int_{J_{e}}s\psi^{e}_{V}(s)\,\mathrm{d}s\right)^{2}-n\int_{J_{e}}s^{2}\psi^{e}_{V}(s)\,\mathrm{d}s-2n\Big((\overset{e}{m}{}^{\infty}_{1})_{11}+(\mathcal{R}^{e,\infty}_{1}(n))_{11}\Big)\int_{J_{e}}s\psi^{e}_{V}(s)\,\mathrm{d}s\\
&+(\overset{e}{m}{}^{\infty}_{2})_{11}+(\mathcal{R}^{e,\infty}_{2}(n))_{11}+(\mathcal{R}^{e,\infty}_{1}(n))_{11}(\overset{e}{m}{}^{\infty}_{1})_{11}+(\mathcal{R}^{e,\infty}_{1}(n))_{12}(\overset{e}{m}{}^{\infty}_{1})_{21},\\
(\mathrm{Y}^{e,\infty}_{2})_{12} &= \mathrm{e}^{n\ell_{e}}\bigg(2n\Big((\overset{e}{m}{}^{\infty}_{1})_{12}+(\mathcal{R}^{e,\infty}_{1}(n))_{12}\Big)\int_{J_{e}}s\psi^{e}_{V}(s)\,\mathrm{d}s+(\overset{e}{m}{}^{\infty}_{2})_{12}+(\mathcal{R}^{e,\infty}_{2}(n))_{12}\\
&+(\mathcal{R}^{e,\infty}_{1}(n))_{11}(\overset{e}{m}{}^{\infty}_{1})_{12}+(\mathcal{R}^{e,\infty}_{1}(n))_{12}(\overset{e}{m}{}^{\infty}_{1})_{22}\bigg),\\
(\mathrm{Y}^{e,\infty}_{2})_{21} &= \mathrm{e}^{-n\ell_{e}}\bigg(-2n\Big((\overset{e}{m}{}^{\infty}_{1})_{21}+(\mathcal{R}^{e,\infty}_{1}(n))_{21}\Big)\int_{J_{e}}s\psi^{e}_{V}(s)\,\mathrm{d}s+(\overset{e}{m}{}^{\infty}_{2})_{21}+(\mathcal{R}^{e,\infty}_{2}(n))_{21}\\
&+(\mathcal{R}^{e,\infty}_{1}(n))_{21}(\overset{e}{m}{}^{\infty}_{1})_{11}+(\mathcal{R}^{e,\infty}_{1}(n))_{22}(\overset{e}{m}{}^{\infty}_{1})_{21}\bigg),\\
(\mathrm{Y}^{e,\infty}_{2})_{22} &= 2n^{2}\left(\int_{J_{e}}s\psi^{e}_{V}(s)\,\mathrm{d}s\right)^{2}+n\int_{J_{e}}s^{2}\psi^{e}_{V}(s)\,\mathrm{d}s+2n\Big((\overset{e}{m}{}^{\infty}_{1})_{22}+(\mathcal{R}^{e,\infty}_{1}(n))_{22}\Big)\int_{J_{e}}s\psi^{e}_{V}(s)\,\mathrm{d}s\\
&+(\overset{e}{m}{}^{\infty}_{2})_{22}+(\mathcal{R}^{e,\infty}_{2}(n))_{22}+(\mathcal{R}^{e,\infty}_{1}(n))_{21}(\overset{e}{m}{}^{\infty}_{1})_{12}+(\mathcal{R}^{e,\infty}_{1}(n))_{22}(\overset{e}{m}{}^{\infty}_{1})_{22}.
\end{aligned}$$

Proof. Let $\mathcal{R}^{e}\colon\mathbb{C}\setminus\widetilde{\Sigma}^{e}_{p}\to\mathrm{SL}_{2}(\mathbb{C})$ be the solution of the RHP $(\mathcal{R}^{e}(z),\upsilon^{e}_{\mathcal{R}}(z),\widetilde{\Sigma}^{e}_{p})$ formulated in Proposition 5.2 with $n\to\infty$ asymptotics given in Lemma 5.3. For $|z|\gg\max_{j=1,\dots,N+1}\{|b^{e}_{j-1}-a^{e}_{j}|\}$, via the expansion $\frac{1}{s-z}=-\sum_{k=0}^{l}\frac{s^{k}}{z^{k+1}}+\frac{s^{l+1}}{z^{l+1}(s-z)}$, $l\in\mathbb{Z}^{+}_{0}$, where $s\in\{b^{e}_{j-1},a^{e}_{j}\}$, $j=1,\dots,N+1$, one obtains the asymptotics for $\mathcal{R}^{e}(z)$ stated in the Proposition.

Let $\overset{e}{m}{}^{\infty}\colon\mathbb{C}\setminus J^{\infty}_{e}\to\mathrm{SL}_{2}(\mathbb{C})$ solve the RHP $(\overset{e}{m}{}^{\infty}(z),J^{\infty}_{e},\overset{e}{\upsilon}{}^{\infty}(z))$ formulated in Lemma 4.3 with (unique) solution given by Lemma 4.5. In order to obtain large-z asymptotics of $\overset{e}{m}{}^{\infty}(z)$, one needs large-$z$ asymptotics of $(\gamma^{e}(z))^{\pm1}$ and $\frac{\boldsymbol{\theta}^{e}(\varepsilon_{1}\boldsymbol{u}^{e}(z)-\frac{n}{2\pi}\boldsymbol{\Omega}^{e}+\varepsilon_{2}\boldsymbol{d}_{e})}{\boldsymbol{\theta}^{e}(\varepsilon_{1}\boldsymbol{u}^{e}(z)+\varepsilon_{2}\boldsymbol{d}_{e})}$, $\varepsilon_{1},\varepsilon_{2}=\pm1$. Consider, say, and without loss of generality, $z\to\infty$ asymptotics for $z\in\mathbb{C}_{+}$, that is, $z\to\infty^{+}$, where, by definition, $\sqrt{\star(z)}:=+\sqrt{\star(z)}$: equivalently, one may consider $z\to\infty$ asymptotics for $z\in\mathbb{C}_{-}$, that is, $z\to\infty^{-}$; however, recalling that $\sqrt{\star(z)}\restriction_{\mathbb{C}_{+}}=-\sqrt{\star(z)}\restriction_{\mathbb{C}_{-}}$, one obtains (in either case, and via the sheet-interchange index) the same $z\to\infty$ asymptotics (for $\overset{e}{m}{}^{\infty}(z)$). Recall the expression for $\gamma^{e}(z)$ given in Lemma 4.4: for $|z|\gg\max_{j=1,\dots,N+1}\{|b^{e}_{j-1}-a^{e}_{j}|\}$, via the expansions $\frac{1}{s-z}=-\sum_{k=0}^{l}\frac{s^{k}}{z^{k+1}}+\frac{s^{l+1}}{z^{l+1}(s-z)}$, $l\in\mathbb{Z}^{+}_{0}$, and $\ln(z-s)=_{|z|\to\infty}\ln(z)-\sum_{k=1}^{\infty}\frac{1}{k}(\frac{s}{z})^{k}$, where $s\in\{b^{e}_{j-1},a^{e}_{j}\}$, $j=1,\dots,N+1$, one shows that

$$(\gamma^{e}(z))^{\pm1}\underset{z\to\infty^{+}}{=}1+\frac{1}{z}\left(\pm\frac{1}{4}\sum_{k=1}^{N+1}(a^{e}_{k}-b^{e}_{k-1})\right)+\frac{1}{z^{2}}\left(\pm\frac{1}{8}\sum_{k=1}^{N+1}\Big((a^{e}_{k})^{2}-(b^{e}_{k-1})^{2}\Big)\right.$$

$$+\frac{1}{32}\left(\sum_{k=1}^{N+1}(a_k^e-b_{k-1}^e)\right)^2\Bigg)+\mathcal{O}\left(\frac{1}{z^3}\right),$$

whence

$$\frac{1}{2}(\gamma^e(z)+(\gamma^e(z))^{-1})\underset{z\to\infty^+}{=}1+\frac{1}{z^2}\left(\frac{1}{32}\left(\sum_{k=1}^{N+1}(a_k^e-b_{k-1}^e)\right)^2\right)+\mathcal{O}\left(\frac{1}{z^3}\right),$$

and

$$\frac{1}{2\mathrm{i}}(\gamma^e(z)-(\gamma^e(z))^{-1})\underset{z\to\infty^+}{=}\frac{1}{z}\left(\frac{1}{4\mathrm{i}}\sum_{k=1}^{N+1}(a_k^e-b_{k-1}^e)\right)+\frac{1}{z^2}\left(\frac{1}{8\mathrm{i}}\sum_{k=1}^{N+1}\Big((a_k^e)^2-(b_{k-1}^e)^2\Big)\right)+\mathcal{O}\left(\frac{1}{z^3}\right).$$

Recall from Lemma 4.5 that $\boldsymbol{u}^e(z):=\int_{a_{N+1}^e}^{z}\boldsymbol{\omega}^e$ $(\in \mathrm{Jac}(\mathcal{Y}_e)$, with $\mathcal{Y}_e:=\{(y,z);\, y^2=R_e(z)\})$, where $\boldsymbol{\omega}^e$, the associated normalised basis of holomorphic one-forms of $\mathcal{Y}_e$, is given by $\boldsymbol{\omega}^e=(\omega_1^e,\omega_2^e,\dots,\omega_N^e)$, with $\omega_j^e:=\sum_{k=1}^{N}c_{jk}^e(\prod_{i=1}^{N+1}(z-b_{i-1}^e)(z-a_i^e))^{-1/2}z^{N-k}\,\mathrm{d}z$, $j=1,\dots,N$, where c_{jk}^e, $j,k=1,\dots,N$, are obtained from Equations (E1) and (E2). Writing

$$\boldsymbol{u}^e(z)=\left(\int_{a_{N+1}^e}^{\infty^+}+\int_{\infty^+}^{z}\right)\boldsymbol{\omega}^e=\boldsymbol{u}_+^e(\infty)+\int_{\infty^+}^{z}\boldsymbol{\omega}^e,$$

where $\boldsymbol{u}_+^e(\infty):=\int_{a_{N+1}^e}^{\infty^+}\boldsymbol{\omega}^e$ (cf. Lemma 4.5), for $|z|\gg\max_{j=1,\dots,N+1}\{|b_{j-1}^e-a_j^e|\}$, via the expansions $\frac{1}{s-z}=-\sum_{k=0}^{l}\frac{s^k}{z^{k+1}}+\frac{s^{l+1}}{z^{l+1}(s-z)}$, $l\in\mathbb{Z}_0^+$, and $\ln(z-s)=_{|z|\to\infty}\ln(z)-\sum_{k=1}^{\infty}\frac{1}{k}(\frac{s}{z})^k$, where $s\in\{b_{k-1}^e,a_k^e\}$, $k=1,\dots,N+1$, one shows that, for $j=1,\dots,N$,

$$\omega_j^e\underset{z\to\infty^+}{=}\frac{c_{j1}^e}{z^2}\,\mathrm{d}z+\frac{(c_{j2}^e+\frac{1}{2}c_{j1}^e\sum_{i=1}^{N+1}(a_i^e+b_{i-1}^e))}{z^3}\,\mathrm{d}z+\mathcal{O}\left(\frac{\mathrm{d}z}{z^4}\right),$$

whence

$$\begin{aligned}\int_{\infty^+}^{z}\omega_j^e&\underset{z\to\infty^+}{=}-\frac{c_{j1}^e}{z}-\frac{\frac{1}{2}(c_{j2}^e+\frac{1}{2}c_{j1}^e\sum_{i=1}^{N+1}(a_i^e+b_{i-1}^e))}{z^2}+\mathcal{O}\left(\frac{1}{z^3}\right)\\&=:-\frac{\widehat{\alpha}_{\infty,j}^e}{z}-\frac{\widehat{\beta}_{\infty,j}^e}{z^2}+\mathcal{O}\left(\frac{1}{z^3}\right).\end{aligned}$$

Defining $\theta_\infty^e(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^e)$, $\alpha_\infty^e(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^e)$, and $\beta_\infty^e(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^e)$, $\varepsilon_1,\varepsilon_2=\pm1$, as in the Proposition, recalling that $\boldsymbol{\omega}^e=(\omega_1^e,\omega_2^e,\dots,\omega_N^e)$, and that the associated $N\times N$ Riemann matrix of $\boldsymbol{\beta}^e$-periods, that is, $\tau^e=(\tau^e)_{i,j=1,\dots,N}:=(\oint_{\boldsymbol{\beta}_j^e}\omega_i^e)_{i,j=1,\dots,N}$, is non-degenerate, symmetric, and $-\mathrm{i}\tau^e$ is positive definite, via the above asymptotic (as $z\to\infty^+$) expansion for $\int_{\infty^+}^{z}\omega_j^e$, $j=1,\dots,N$, one shows that

$$\frac{\boldsymbol{\theta}^e(\varepsilon_1\boldsymbol{u}^e(z)-\frac{n}{2\pi}\boldsymbol{\Omega}^e+\varepsilon_2\boldsymbol{d}_e)}{\boldsymbol{\theta}^e(\varepsilon_1\boldsymbol{u}^e(z)+\varepsilon_2\boldsymbol{d}_e)}\underset{z\to\infty^+}{=}\Theta_0^e+\frac{1}{z}\Theta_1^e+\frac{1}{z^2}\Theta_2^e+\mathcal{O}\left(\frac{1}{z^3}\right),$$

where

$$\begin{aligned}\Theta_0^e&:=\frac{\theta_\infty^e(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^e)}{\theta_\infty^e(\varepsilon_1,\varepsilon_2,\vec{\boldsymbol{0}})},\\ \Theta_1^e&:=\frac{\theta_\infty^e(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^e)\alpha_\infty^e(\varepsilon_1,\varepsilon_2,\vec{\boldsymbol{0}})-\alpha_\infty^e(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^e)\theta_\infty^e(\varepsilon_1,\varepsilon_2,\vec{\boldsymbol{0}})}{(\theta_\infty^e(\varepsilon_1,\varepsilon_2,\vec{\boldsymbol{0}}))^2},\\ \Theta_2^e&:=\Big(\theta_\infty^e(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^e)\Big(\beta_\infty^e(\varepsilon_1,\varepsilon_2,\vec{\boldsymbol{0}})\theta_\infty^e(\varepsilon_1,\varepsilon_2,\vec{\boldsymbol{0}})+(\alpha_\infty^e(\varepsilon_1,\varepsilon_2,\vec{\boldsymbol{0}}))^2\Big)-\alpha_\infty^e(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^e)\\&\quad\times\alpha_\infty^e(\varepsilon_1,\varepsilon_2,\vec{\boldsymbol{0}})\theta_\infty^e(\varepsilon_1,\varepsilon_2,\vec{\boldsymbol{0}})-\beta_\infty^e(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^e)(\theta_\infty^e(\varepsilon_1,\varepsilon_2,\vec{\boldsymbol{0}}))^2\Big)(\theta_\infty^e(\varepsilon_1,\varepsilon_2,\vec{\boldsymbol{0}}))^{-3},\end{aligned}$$

with $\vec{\boldsymbol{0}}:=(0,0,\dots,0)^{\mathrm{T}}$ $(\in\mathbb{R}^N)$. Via the above asymptotic (as $z\to\infty^+$) expansions for $\frac{1}{2}(\gamma^e(z)+(\gamma^e(z))^{-1})$, $\frac{1}{2\mathrm{i}}(\gamma^e(z)-(\gamma^e(z))^{-1})$, and $\frac{\boldsymbol{\theta}^e(\varepsilon_1\boldsymbol{u}^e(z)-\frac{n}{2\pi}\boldsymbol{\Omega}^e+\varepsilon_2\boldsymbol{d}_e)}{\boldsymbol{\theta}^e(\varepsilon_1\boldsymbol{u}^e(z)+\varepsilon_2\boldsymbol{d}_e)}$, one arrives at, upon recalling the expression for $\overset{e}{m}{}^\infty(z)$ given in Lemma 4.5, the asymptotic expansion for $\overset{e}{m}{}^\infty(z)$ stated in the Proposition.

Let $\overset{e}{\mathrm{Y}}\colon \mathbb{C}\setminus\mathbb{R}\to\mathrm{SL}_2(\mathbb{C})$ be the (unique) solution of **RHP1**, that is, $(\overset{e}{\mathrm{Y}}(z),\mathrm{I}+\exp(-n\widetilde{V}(z))\sigma_+,\mathbb{R})$. Recall, also, that, for $z\in\Upsilon_1^e\cup\Upsilon_2^e$ (Figure 7),

$$\overset{e}{\mathrm{Y}}(z)=\mathrm{e}^{\frac{n\ell_e}{2}\operatorname{ad}(\sigma_3)}\mathcal{R}^e(z)\overset{e}{m}{}^{\infty}(z)\mathrm{e}^{n(g^e(z)+\int_{J_e}\ln(s)\psi_V^e(s)\,\mathrm{d}s)\sigma_3}:$$

consider, say, and without loss of generality, large-z asymptotics for $\overset{e}{\mathrm{Y}}(z)$ for $z\in\Upsilon_1^e$. Recalling the definition of $g^e(z)$ given in Lemma 3.4, that is, $g^e(z):=\int_{J_e}\ln((z-s)^2(zs)^{-1})\psi_V^e(s)\,\mathrm{d}s$, $z\in\mathbb{C}\setminus(-\infty,\max\{0,a_{N+1}^e\})$, for $|z|\gg\max_{j=1,\dots,N+1}\{|b_{j-1}^e-a_j^e|\}$, in particular, $|s/z|\ll 1$ with $s\in J_e$, and noting that $\int_{J_e}\psi_V^e(s)\,\mathrm{d}s=1$ and $\int_{J_e}s^m\psi_V^e(s)\,\mathrm{d}s<\infty$, $m\in\mathbb{N}$, via the expansions $\frac{1}{s-z}=-\sum_{k=0}^{l}\frac{s^k}{z^{k+1}}+\frac{s^{l+1}}{z^{l+1}(s-z)}$, $l\in\mathbb{Z}_0^+$, and $\ln(z-s)=_{|z|\to\infty}\ln(z)-\sum_{k=1}^{\infty}\frac{1}{k}(\frac{s}{z})^k$, one shows that

$$g^e(z)\underset{z\to\infty}{=}\ln(z)-\int_{J_e}\ln(s)\psi_V^e(s)\,\mathrm{d}s+\frac{1}{z}\left(-2\int_{J_e}s\psi_V^e(s)\,\mathrm{d}s\right)+\frac{1}{z^2}\left(-\int_{J_e}s^2\psi_V^e(s)\,\mathrm{d}s\right)+O\left(\frac{1}{z^3}\right)$$

(explicit expressions for $\int_{J_e}s^k\psi_V^e(s)\,\mathrm{d}s$, $k=1,2$, are given in Remark 3.2): using the asymptotic (as $z\to\infty$) expansions for $g^e(z)$, $\mathcal{R}^e(z)$, and $\overset{e}{m}{}^{\infty}(z)$ derived above, upon recalling the formula for $\overset{e}{\mathrm{Y}}(z)$, one arrives at, after a matrix-multiplication argument, the asymptotic expansion for $\overset{e}{\mathrm{Y}}(z)z^{-n\sigma_3}$ stated in the Proposition. □

Proposition 5.4. *Let $\overset{e}{\mathrm{Y}}\colon\mathbb{C}\setminus\mathbb{R}\to\mathrm{SL}_2(\mathbb{C})$ be the solution of* **RHP1** *with z $(\in\mathbb{C}\setminus\mathbb{R})\to\infty$ asymptotics given in Proposition* 5.3. *Then,*

$$\xi_n^{(2n)}=\frac{1}{\|\boldsymbol{\pi}_{2n}(\cdot)\|_{\mathcal{L}}}=\sqrt{\frac{H_{2n}^{(-2n)}}{H_{2n+1}^{(-2n)}}}=\left(-\frac{1}{2\pi\mathrm{i}(\mathrm{Y}_1^{e,\infty})_{12}}\right)^{1/2}\quad(>0),$$

where $(\mathrm{Y}_1^{e,\infty})_{12}=\mathrm{e}^{n\ell_e}\left((\overset{e}{m}{}_1^{\infty})_{12}+(\mathcal{R}_1^{e,\infty}(n))_{12}\right)$, with $(\overset{e}{m}{}_1^{\infty})_{12}$ and $(\mathcal{R}_1^{e,\infty}(n))_{12}$ given in Proposition 5.3.

Proof. Recall from Lemma 2.2.1 that $\boldsymbol{\pi}_{2n}(z):=(\overset{e}{\mathrm{Y}}(z))_{11}$ and $(\overset{e}{\mathrm{Y}}(z))_{12}=\int_{\mathbb{R}}\frac{\boldsymbol{\pi}_{2n}(s)\exp(-n\widetilde{V}(s))}{s-z}\frac{\mathrm{d}s}{2\pi\mathrm{i}}$. Using (for $|s/z|\ll 1$) the expansion $\frac{1}{s-z}=-\sum_{k=0}^{l}\frac{s^k}{z^{k+1}}+\frac{s^{l+1}}{z^{l+1}(s-z)}$, $l\in\mathbb{Z}_0^+$, and recalling that $\langle\boldsymbol{\pi}_{2n},z^j\rangle_{\mathcal{L}}=0$, $j=-n,\dots,n-1$, and $\phi_{2n}(z)=\xi_n^{(2n)}\boldsymbol{\pi}_{2n}(z)$, one proceeds as follows:

$$\begin{aligned}
\left(\overset{e}{\mathrm{Y}}(z)\right)_{12}&\underset{\substack{z\to\infty\\ z\in\mathbb{C}\setminus\mathbb{R}}}{=}-\frac{1}{z}\int_{\mathbb{R}}\boldsymbol{\pi}_{2n}(s)\left(1+\frac{s}{z}+\dots+\frac{s^{n-1}}{z^{n-1}}+\frac{s^n}{z^n}+\cdots\right)\mathrm{e}^{-n\widetilde{V}(s)}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\\
&\underset{\substack{z\to\infty\\ z\in\mathbb{C}\setminus\mathbb{R}}}{=}-\frac{1}{z}\int_{\mathbb{R}}\boldsymbol{\pi}_{2n}(s)\left(\frac{s^n}{z^n}\right)\mathrm{e}^{-n\widetilde{V}(s)}\frac{\mathrm{d}s}{2\pi\mathrm{i}}+O\left(\frac{1}{z^{n+2}}\right)\\
&\underset{\substack{z\to\infty\\ z\in\mathbb{C}\setminus\mathbb{R}}}{=}-\frac{z^{-(n+1)}}{\xi_n^{(2n)}}\int_{\mathbb{R}}\underbrace{\xi_n^{(2n)}\boldsymbol{\pi}_{2n}(s)}_{=\phi_{2n}(s)}\frac{\mathrm{e}^{-n\widetilde{V}(s)}}{\xi_n^{(2n)}}\underbrace{\left(\xi_n^{(2n)}s^n+\dots+\frac{\xi_{-n}^{(2n)}}{s^n}\right)}_{=\phi_{2n}(s)}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\\
&\quad+O\left(z^{-(n+2)}\right)\\
&\underset{\substack{z\to\infty\\ z\in\mathbb{C}\setminus\mathbb{R}}}{=}-\frac{z^{-(n+1)}}{2\pi\mathrm{i}(\xi_n^{(2n)})^2}\underbrace{\int_{\mathbb{R}}\phi_{2n}(s)\phi_{2n}(s)\mathrm{e}^{-n\widetilde{V}(s)}\,\mathrm{d}s}_{=1}+O\left(z^{-(n+2)}\right)\Rightarrow
\end{aligned}$$

$$\left(\overset{e}{\mathrm{Y}}(z)z^{-n\sigma_3}\right)_{12}\underset{\substack{z\to\infty\\ z\in\mathbb{C}\setminus\mathbb{R}}}{=}\frac{1}{z}\left(-\frac{1}{2\pi\mathrm{i}(\xi_n^{(2n)})^2}\right)+O\left(\frac{1}{z^2}\right);$$

but, noting from Proposition 5.3 that

$$\left(\overset{e}{\mathrm{Y}}(z)z^{-n\sigma_3}\right)_{12}\underset{\substack{z\to\infty\\ z\in\mathbb{C}\setminus\mathbb{R}}}{=}\frac{1}{z}\left(\mathrm{Y}_1^{e,\infty}\right)_{12}+O\left(\frac{1}{z^2}\right),$$

upon equating the above two asymptotic expansions for $(\overset{e}{Y}(z)z^{-n\sigma_3})_{12}$, one arrives at the result stated in the Proposition. □

Using the results of Propositions 5.3 and 5.4, one obtains the $n \to \infty$ asymptotics for $\xi_n^{(2n)}$ and $\phi_{2n}(z)$ (in the entire complex plane) stated in Theorem 2.3.2.

Small-z asymptotics for $\overset{e}{Y}(z)$ are given in the Appendix (see Lemma A.1): these latter asymptotics are necessary for the results of [52].

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Appendix: Small-z Asymptotics for $\overset{e}{\mathrm{Y}}(z)$

Even though the results of Lemma A.1 below, namely, small-z asymptotics (as $(\mathbb{C}\setminus\mathbb{R}\ni)\ z\to 0$) of $\overset{e}{\mathrm{Y}}(z)$, are not necessary in order to prove Theorems 2.3.1 and 2.3.2, they are essential for the results of [52], related to asymptotics of the coefficients of the system of three- and five-term recurrence relations and the corresponding Laurent-Jacobi matrices (cf. Section 1). For the sake of completeness, therefore, and in order to eschew any duplication of the analysis of this paper, $(\mathbb{C}\setminus\mathbb{R}\ni)\ z\to 0$ asymptotics for $\overset{e}{\mathrm{Y}}(z)$ are presented here.

Lemma A.1. *Let $\mathcal{R}^e\colon \mathbb{C}\setminus\widetilde{\Sigma}_p^e\to\mathrm{SL}_2(\mathbb{C})$ be the solution of the* RHP $(\mathcal{R}^e(z), v_{\mathcal{R}}^e(z), \widetilde{\Sigma}_p^e)$ *formulated in Proposition* 5.2 *with $n\to\infty$ asymptotics given in Lemma* 5.3. *Then,*

$$\mathcal{R}^e(z) \underset{z\to 0}{=} \mathrm{I}+\mathcal{R}_0^{e,0}(n)+\mathcal{R}_1^{e,0}(n)z+\mathcal{R}_2^{e,0}(n)z^2+\mathcal{O}(z^3),$$

where, for $k=1,2,3$,

$$\mathcal{R}_{k-1}^{e,0}(n):=\int_{\Sigma_{\circlearrowright}^e} s^{-k}w_+^{\Sigma_{\circlearrowright}^e}(s)\,\frac{\mathrm{d}s}{2\pi\mathrm{i}}=-\sum_{j=1}^{N+1}\sum_{q\in\{b_{j-1}^e,a_j^e\}}\mathrm{Res}\Big(z^{-k}w_+^{\Sigma_{\circlearrowright}^e}(z);q\Big),$$

with, in particular,

$$\begin{aligned}\mathcal{R}_{k-1}^{e,0}(n) \underset{n\to\infty}{=} \frac{1}{n}\sum_{j=1}^{N+1}&\left(\frac{(\mathcal{A}^e(b_{j-1}^e)(\widehat{\alpha}_1^e(b_{j-1}^e)+k(b_{j-1}^e)^{-1}\widehat{\alpha}_0^e(b_{j-1}^e))-\mathcal{B}^e(b_{j-1}^e)\widehat{\alpha}_0^e(b_{j-1}^e))}{(b_{j-1}^e)^k(\widehat{\alpha}_0^e(b_{j-1}^e))^2}\right.\\ &+\left.\frac{(\mathcal{A}^e(a_j^e)(\widehat{\alpha}_1^e(a_j^e)+k(a_j^e)^{-1}\widehat{\alpha}_0^e(a_j^e))-\mathcal{B}^e(a_j^e)\widehat{\alpha}_0^e(a_j^e))}{(a_j^e)^k(\widehat{\alpha}_0^e(a_j^e))^2}\right)+\mathcal{O}\Big(\frac{1}{n^2}\Big),\end{aligned}$$

and all the parameters defined in Lemma 5.3.

Let $\overset{e}{m}{}^\infty\colon \mathbb{C}\setminus J_e^\infty\to\mathrm{SL}_2(\mathbb{C})$ solve the RHP $(\overset{e}{m}{}^\infty(z), J_e^\infty, \overset{e}{v}{}^\infty(z))$ *formulated in Lemma* 4.3 *with (unique) solution given by Lemma* 4.5. *For $\varepsilon_1,\varepsilon_2=\pm 1$, set*

$$\theta_0^e(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^e):=\boldsymbol{\theta}^e(\varepsilon_1\boldsymbol{u}_+^e(0)-\tfrac{n}{2\pi}\boldsymbol{\Omega}^e+\varepsilon_2\boldsymbol{d}_e),$$

where $\boldsymbol{u}_+^e(0)=\int_{a_{N+1}^e}^{0^+}\boldsymbol{\omega}^e$ $(0^+\in\mathbb{C}_+)$,

$$\widetilde{\alpha}_0^e(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^e):=2\pi\mathrm{i}\varepsilon_1\sum_{m\in\mathbb{Z}^N}(m,\widehat{\boldsymbol{\alpha}}_0^e)\mathrm{e}^{2\pi\mathrm{i}(m,\varepsilon_1\boldsymbol{u}_+^e(0)-\frac{n}{2\pi}\boldsymbol{\Omega}^e+\varepsilon_2\boldsymbol{d}_e)+\pi\mathrm{i}(m,\tau^e m)},$$

where $\widehat{\boldsymbol{\alpha}}_0^e=(\widehat{\alpha}_{0,1}^e,\widehat{\alpha}_{0,2}^e,\dots,\widehat{\alpha}_{0,N}^e)$, with $\widehat{\alpha}_{0,j}^e:=(-1)^{\mathcal{N}_+}(\prod_{i=1}^{N+1}|b_{i-1}^e a_i^e|)^{-1/2}c_{jN}^e$, $j=1,\dots,N$, where $\mathcal{N}_+\in\{0,\dots,N+1\}$ is the number of bands to the right of $z=0$,

$$\beta_0^e(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^e):=2\pi\sum_{m\in\mathbb{Z}^N}\Big(\mathrm{i}\varepsilon_1(m,\widehat{\boldsymbol{\beta}}_0^e)-\pi(m,\widehat{\boldsymbol{\alpha}}_0^e)^2\Big)\mathrm{e}^{2\pi\mathrm{i}(m,\varepsilon_1\boldsymbol{u}_+^e(0)-\frac{n}{2\pi}\boldsymbol{\Omega}^e+\varepsilon_2\boldsymbol{d}_e)+\pi\mathrm{i}(m,\tau^e m)},$$

where $\widehat{\boldsymbol{\beta}}_0^e=(\widehat{\beta}_{0,1}^e,\widehat{\beta}_{0,2}^e,\dots,\widehat{\beta}_{0,N}^e)$, with $\widehat{\beta}_{0,j}^e:=\frac{1}{2}(-1)^{\mathcal{N}_+}(\prod_{i=1}^{N+1}|b_{i-1}^e a_i^e|)^{-1/2}(c_{jN-1}^e+\frac{1}{2}c_{jN}^e\sum_{k=1}^{N+1}((a_k^e)^{-1}+(b_{k-1}^e)^{-1}))$, $j=1,\dots,N$, where c_{jN}^e, c_{jN-1}^e, $j=1,\dots,N$, are obtained from Equations (E1) *and* (E2). *Set $\gamma_0^e:=\gamma^e(0)=(\prod_{k=1}^{N+1}b_{k-1}^e(a_k^e)^{-1})^{1/4}$ (>0). Then,*

$$\overset{e}{m}{}^\infty(z)\underset{z\to 0}{=}\overset{e}{m}{}_0^0+z\overset{e}{m}{}_1^0+z^2\overset{e}{m}{}_2^0+\mathcal{O}(z^3),$$

where

$$(\overset{e}{m}{}_0^0)_{11}=\frac{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\infty)+\boldsymbol{d}_e)}{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\infty)-\frac{n}{2\pi}\boldsymbol{\Omega}^e+\boldsymbol{d}_e)}\left(\frac{\gamma_0^e+(\gamma_0^e)^{-1}}{2}\right)\frac{\theta_0^e(1,1,\boldsymbol{\Omega}^e)}{\theta_0^e(1,1,\vec{\boldsymbol{0}})},$$

$$
\begin{aligned}
(\overset{e}{m}{}_0^0)_{12} &= -\frac{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\infty)+\boldsymbol{d}_e)}{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\infty)-\frac{n}{2\pi}\boldsymbol{\Omega}^e+\boldsymbol{d}_e)}\left(\frac{\gamma_0^e-(\gamma_0^e)^{-1}}{2\mathrm{i}}\right)\frac{\theta_0^e(-1,1,\boldsymbol{\Omega}^e)}{\theta_0^e(-1,1,\vec{\boldsymbol{0}})},\\
(\overset{e}{m}{}_0^0)_{21} &= \frac{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\infty)+\boldsymbol{d}_e)}{\boldsymbol{\theta}^e(-\boldsymbol{u}_+^e(\infty)-\frac{n}{2\pi}\boldsymbol{\Omega}^e-\boldsymbol{d}_e)}\left(\frac{\gamma_0^e-(\gamma_0^e)^{-1}}{2\mathrm{i}}\right)\frac{\theta_0^e(1,-1,\boldsymbol{\Omega}^e)}{\theta_0^e(1,-1,\vec{\boldsymbol{0}})},\\
(\overset{e}{m}{}_0^0)_{22} &= \frac{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\infty)+\boldsymbol{d}_e)}{\boldsymbol{\theta}^e(-\boldsymbol{u}_+^e(\infty)-\frac{n}{2\pi}\boldsymbol{\Omega}^e-\boldsymbol{d}_e)}\left(\frac{\gamma_0^e+(\gamma_0^e)^{-1}}{2}\right)\frac{\theta_0^e(-1,-1,\boldsymbol{\Omega}^e)}{\theta_0^e(-1,-1,\vec{\boldsymbol{0}})},\\
(\overset{e}{m}{}_1^0)_{11} &= \frac{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\infty)+\boldsymbol{d}_e)}{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\infty)-\frac{n}{2\pi}\boldsymbol{\Omega}^e+\boldsymbol{d}_e)}\left(\left(\frac{\widetilde{\alpha}_0^e(1,1,\boldsymbol{\Omega}^e)\theta_0^e(1,1,\vec{\boldsymbol{0}})-\widetilde{\alpha}_0^e(1,1,\vec{\boldsymbol{0}})\theta_0^e(1,1,\boldsymbol{\Omega}^e)}{(\theta_0^e(1,1,\vec{\boldsymbol{0}}))^2}\right)\right.\\
&\quad\left.\times\left(\frac{\gamma_0^e+(\gamma_0^e)^{-1}}{2}\right)+\left(\frac{\gamma_0^e-(\gamma_0^e)^{-1}}{8}\right)\left(\sum_{k=1}^{N+1}\left(\frac{1}{a_k^e}-\frac{1}{b_{k-1}^e}\right)\right)\frac{\theta_0^e(1,1,\boldsymbol{\Omega}^e)}{\theta_0^e(1,1,\vec{\boldsymbol{0}})}\right),\\
(\overset{e}{m}{}_1^0)_{12} &= -\frac{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\infty)+\boldsymbol{d}_e)}{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\infty)-\frac{n}{2\pi}\boldsymbol{\Omega}^e+\boldsymbol{d}_e)}\left(\left(\frac{\widetilde{\alpha}_0^e(-1,1,\boldsymbol{\Omega}^e)\theta_0^e(-1,1,\vec{\boldsymbol{0}})-\widetilde{\alpha}_0^e(-1,1,\vec{\boldsymbol{0}})\theta_0^e(-1,1,\boldsymbol{\Omega}^e)}{(\theta_0^e(-1,1,\vec{\boldsymbol{0}}))^2}\right)\right.\\
&\quad\left.\times\left(\frac{\gamma_0^e-(\gamma_0^e)^{-1}}{2\mathrm{i}}\right)+\left(\frac{\gamma_0^e+(\gamma_0^e)^{-1}}{8\mathrm{i}}\right)\left(\sum_{k=1}^{N+1}\left(\frac{1}{a_k^e}-\frac{1}{b_{k-1}^e}\right)\right)\frac{\theta_0^e(-1,1,\boldsymbol{\Omega}^e)}{\theta_0^e(-1,1,\vec{\boldsymbol{0}})}\right),\\
(\overset{e}{m}{}_1^0)_{21} &= \frac{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\infty)+\boldsymbol{d}_e)}{\boldsymbol{\theta}^e(-\boldsymbol{u}_+^e(\infty)-\frac{n}{2\pi}\boldsymbol{\Omega}^e-\boldsymbol{d}_e)}\left(\left(\frac{\widetilde{\alpha}_0^e(1,-1,\boldsymbol{\Omega}^e)\theta_0^e(1,-1,\vec{\boldsymbol{0}})-\widetilde{\alpha}_0^e(1,-1,\vec{\boldsymbol{0}})\theta_0^e(1,-1,\boldsymbol{\Omega}^e)}{(\theta_0^e(1,-1,\vec{\boldsymbol{0}}))^2}\right)\right.\\
&\quad\left.\times\left(\frac{\gamma_0^e-(\gamma_0^e)^{-1}}{2\mathrm{i}}\right)+\left(\frac{\gamma_0^e+(\gamma_0^e)^{-1}}{8\mathrm{i}}\right)\left(\sum_{k=1}^{N+1}\left(\frac{1}{a_k^e}-\frac{1}{b_{k-1}^e}\right)\right)\frac{\theta_0^e(1,-1,\boldsymbol{\Omega}^e)}{\theta_0^e(1,-1,\vec{\boldsymbol{0}})}\right),\\
(\overset{e}{m}{}_1^0)_{22} &= \frac{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\infty)+\boldsymbol{d}_e)}{\boldsymbol{\theta}^e(-\boldsymbol{u}_+^e(\infty)-\frac{n}{2\pi}\boldsymbol{\Omega}^e-\boldsymbol{d}_e)}\left(\left(\frac{\widetilde{\alpha}_0^e(-1,-1,\boldsymbol{\Omega}^e)\theta_0^e(-1,-1,\vec{\boldsymbol{0}})-\widetilde{\alpha}_0^e(-1,-1,\vec{\boldsymbol{0}})\theta_0^e(-1,-1,\boldsymbol{\Omega}^e)}{(\theta_0^e(-1,-1,\vec{\boldsymbol{0}}))^2}\right)\right.\\
&\quad\left.\times\left(\frac{\gamma_0^e+(\gamma_0^e)^{-1}}{2}\right)+\left(\frac{\gamma_0^e-(\gamma_0^e)^{-1}}{8}\right)\left(\sum_{k=1}^{N+1}\left(\frac{1}{a_k^e}-\frac{1}{b_{k-1}^e}\right)\right)\frac{\theta_0^e(-1,-1,\boldsymbol{\Omega}^e)}{\theta_0^e(-1,-1,\vec{\boldsymbol{0}})}\right),\\
(\overset{e}{m}{}_2^0)_{11} &= \frac{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\infty)+\boldsymbol{d}_e)}{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\infty)-\frac{n}{2\pi}\boldsymbol{\Omega}^e+\boldsymbol{d}_e)}\Big(\Big(\theta_0^e(1,1,\boldsymbol{\Omega}^e)\Big((\widetilde{\alpha}_0^e(1,1,\vec{\boldsymbol{0}}))^2-\beta_0^e(1,1,\vec{\boldsymbol{0}})\theta_0^e(1,1,\vec{\boldsymbol{0}})\Big)\\
&\quad-\widetilde{\alpha}_0^e(1,1,\boldsymbol{\Omega}^e)\widetilde{\alpha}_0^e(1,1,\vec{\boldsymbol{0}})\theta_0^e(1,1,\vec{\boldsymbol{0}})+\beta_0^e(1,1,\boldsymbol{\Omega}^e)(\theta_0^e(1,1,\vec{\boldsymbol{0}}))^2\Big)\tfrac{1}{(\theta_0^e(1,1,\vec{\boldsymbol{0}}))^3}\\
&\quad\times\left(\frac{\gamma_0^e+(\gamma_0^e)^{-1}}{2}\right)+\left(\frac{\widetilde{\alpha}_0^e(1,1,\boldsymbol{\Omega}^e)\theta_0^e(1,1,\vec{\boldsymbol{0}})-\widetilde{\alpha}_0^e(1,1,\vec{\boldsymbol{0}})\theta_0^e(1,1,\boldsymbol{\Omega}^e)}{(\theta_0^e(1,1,\vec{\boldsymbol{0}}))^2}\right)\\
&\quad\times\left(\sum_{k=1}^{N+1}\left(\frac{1}{a_k^e}-\frac{1}{b_{k-1}^e}\right)\right)\left(\frac{\gamma_0^e-(\gamma_0^e)^{-1}}{8}\right)+\left(\left(\frac{\gamma_0^e-(\gamma_0^e)^{-1}}{16}\right)\sum_{k=1}^{N+1}\left(\frac{1}{(a_k^e)^2}-\frac{1}{(b_{k-1}^e)^2}\right)\right.\\
&\quad\left.+\left(\frac{\gamma_0^e+(\gamma_0^e)^{-1}}{64}\right)\left(\sum_{k=1}^{N+1}\left(\frac{1}{a_k^e}-\frac{1}{b_{k-1}^e}\right)\right)^2\right)\frac{\theta_0^e(1,1,\boldsymbol{\Omega}^e)}{\theta_0^e(1,1,\vec{\boldsymbol{0}})}\Bigg),\\
(\overset{e}{m}{}_2^0)_{12} &= -\frac{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\infty)+\boldsymbol{d}_e)}{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\infty)-\frac{n}{2\pi}\boldsymbol{\Omega}^e+\boldsymbol{d}_e)}\Big(\Big(\theta_0^e(-1,1,\boldsymbol{\Omega}^e)\Big((\widetilde{\alpha}_0^e(-1,1,\vec{\boldsymbol{0}}))^2-\beta_0^e(-1,1,\vec{\boldsymbol{0}})\theta_0^e(-1,1,\vec{\boldsymbol{0}})\Big)\\
&\quad-\widetilde{\alpha}_0^e(-1,1,\boldsymbol{\Omega}^e)\widetilde{\alpha}_0^e(-1,1,\vec{\boldsymbol{0}})\theta_0^e(-1,1,\vec{\boldsymbol{0}})+\beta_0^e(-1,1,\boldsymbol{\Omega}^e)(\theta_0^e(-1,1,\vec{\boldsymbol{0}}))^2\Big)\tfrac{1}{(\theta_0^e(-1,1,\vec{\boldsymbol{0}}))^3}\\
&\quad\times\left(\frac{\gamma_0^e-(\gamma_0^e)^{-1}}{2\mathrm{i}}\right)+\left(\frac{\widetilde{\alpha}_0^e(-1,1,\boldsymbol{\Omega}^e)\theta_0^e(-1,1,\vec{\boldsymbol{0}})-\widetilde{\alpha}_0^e(-1,1,\vec{\boldsymbol{0}})\theta_0^e(-1,1,\boldsymbol{\Omega}^e)}{(\theta_0^e(-1,1,\vec{\boldsymbol{0}}))^2}\right)\\
&\quad\times\left(\sum_{k=1}^{N+1}\left(\frac{1}{a_k^e}-\frac{1}{b_{k-1}^e}\right)\right)\left(\frac{\gamma_0^e+(\gamma_0^e)^{-1}}{8\mathrm{i}}\right)+\left(\left(\frac{\gamma_0^e+(\gamma_0^e)^{-1}}{16\mathrm{i}}\right)\sum_{k=1}^{N+1}\left(\frac{1}{(a_k^e)^2}-\frac{1}{(b_{k-1}^e)^2}\right)\right.
\end{aligned}
$$

$$
\qquad + \bigg(\frac{\gamma_0^e-(\gamma_0^e)^{-1}}{64\mathrm{i}}\bigg)\bigg(\sum_{k=1}^{N+1}\bigg(\frac{1}{a_k^e}-\frac{1}{b_{k-1}^e}\bigg)\bigg)^2\bigg)\frac{\theta_0^e(-1,1,\boldsymbol{\Omega}^e)}{\theta_0^e(-1,1,\vec{\mathbf{0}})}\bigg),
$$

$$
(\overset{e}{m}{}^{0}_{2})_{21} = \frac{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\infty)+\boldsymbol{d}_e)}{\boldsymbol{\theta}^e(-\boldsymbol{u}_+^e(\infty)-\frac{n}{2\pi}\boldsymbol{\Omega}^e-\boldsymbol{d}_e)}\Big(\Big(\theta_0^e(1,-1,\boldsymbol{\Omega}^e)\Big((\widetilde{\alpha}_0^e(1,-1,\vec{\mathbf{0}}))^2-\beta_0^e(1,-1,\vec{\mathbf{0}})\theta_0^e(1,-1,\vec{\mathbf{0}})\Big)
$$

$$
\qquad - \widetilde{\alpha}_0^e(1,-1,\boldsymbol{\Omega}^e)\widetilde{\alpha}_0^e(1,-1,\vec{\mathbf{0}})\theta_0^e(1,-1,\vec{\mathbf{0}})+\beta_0^e(1,-1,\boldsymbol{\Omega}^e)(\theta_0^e(1,-1,\vec{\mathbf{0}}))^2\Big)\frac{1}{(\theta_0^e(1,-1,\vec{\mathbf{0}}))^3}
$$

$$
\qquad \times\bigg(\frac{\gamma_0^e-(\gamma_0^e)^{-1}}{2\mathrm{i}}\bigg)+\bigg(\frac{\widetilde{\alpha}_0^e(1,-1,\boldsymbol{\Omega}^e)\theta_0^e(1,-1,\vec{\mathbf{0}})-\widetilde{\alpha}_0^e(1,-1,\vec{\mathbf{0}})\theta_0^e(1,-1,\boldsymbol{\Omega}^e)}{(\theta_0^e(1,-1,\vec{\mathbf{0}}))^2}\bigg)
$$

$$
\qquad \times\bigg(\sum_{k=1}^{N+1}\bigg(\frac{1}{a_k^e}-\frac{1}{b_{k-1}^e}\bigg)\bigg)\bigg(\frac{\gamma_0^e+(\gamma_0^e)^{-1}}{8\mathrm{i}}\bigg)+\bigg(\bigg(\frac{\gamma_0^e+(\gamma_0^e)^{-1}}{16\mathrm{i}}\bigg)\sum_{k=1}^{N+1}\bigg(\frac{1}{(a_k^e)^2}-\frac{1}{(b_{k-1}^e)^2}\bigg)
$$

$$
\qquad + \bigg(\frac{\gamma_0^e-(\gamma_0^e)^{-1}}{64\mathrm{i}}\bigg)\bigg(\sum_{k=1}^{N+1}\bigg(\frac{1}{a_k^e}-\frac{1}{b_{k-1}^e}\bigg)\bigg)^2\bigg)\frac{\theta_0^e(1,-1,\boldsymbol{\Omega}^e)}{\theta_0^e(1,-1,\vec{\mathbf{0}})}\bigg),
$$

$$
(\overset{e}{m}{}^{0}_{2})_{22} = \frac{\boldsymbol{\theta}^e(\boldsymbol{u}_+^e(\infty)+\boldsymbol{d}_e)}{\boldsymbol{\theta}^e(-\boldsymbol{u}_+^e(\infty)-\frac{n}{2\pi}\boldsymbol{\Omega}^e-\boldsymbol{d}_e)}\Big(\Big(\theta_0^e(-1,-1,\boldsymbol{\Omega}^e)\Big((\widetilde{\alpha}_0^e(-1,-1,\vec{\mathbf{0}}))^2-\beta_0^e(-1,-1,\vec{\mathbf{0}})\theta_0^e(-1,-1,\vec{\mathbf{0}})\Big)
$$

$$
\qquad - \widetilde{\alpha}_0^e(-1,-1,\boldsymbol{\Omega}^e)\widetilde{\alpha}_0^e(-1,-1,\vec{\mathbf{0}})\theta_0^e(-1,-1,\vec{\mathbf{0}})+\beta_0^e(-1,-1,\boldsymbol{\Omega}^e)(\theta_0^e(-1,-1,\vec{\mathbf{0}}))^2\Big)\frac{1}{(\theta_0^e(-1,-1,\vec{\mathbf{0}}))^3}
$$

$$
\qquad \times\bigg(\frac{\gamma_0^e+(\gamma_0^e)^{-1}}{2}\bigg)+\bigg(\frac{\widetilde{\alpha}_0^e(-1,-1,\boldsymbol{\Omega}^e)\theta_0^e(-1,-1,\vec{\mathbf{0}})-\widetilde{\alpha}_0^e(-1,-1,\vec{\mathbf{0}})\theta_0^e(-1,-1,\boldsymbol{\Omega}^e)}{(\theta_0^e(-1,-1,\vec{\mathbf{0}}))^2}\bigg)
$$

$$
\qquad \times\bigg(\sum_{k=1}^{N+1}\bigg(\frac{1}{a_k^e}-\frac{1}{b_{k-1}^e}\bigg)\bigg)\bigg(\frac{\gamma_0^e-(\gamma_0^e)^{-1}}{8}\bigg)+\bigg(\bigg(\frac{\gamma_0^e-(\gamma_0^e)^{-1}}{16}\bigg)\sum_{k=1}^{N+1}\bigg(\frac{1}{(a_k^e)^2}-\frac{1}{(b_{k-1}^e)^2}\bigg)
$$

$$
\qquad + \bigg(\frac{\gamma_0^e+(\gamma_0^e)^{-1}}{64}\bigg)\bigg(\sum_{k=1}^{N+1}\bigg(\frac{1}{a_k^e}-\frac{1}{b_{k-1}^e}\bigg)\bigg)^2\bigg)\frac{\theta_0^e(-1,-1,\boldsymbol{\Omega}^e)}{\theta_0^e(-1,-1,\vec{\mathbf{0}})}\bigg),
$$

with $(\star)_{ij}$, $i,j=1,2$, *denoting the* $(i\,j)$*-element of* $\star$, *and* $\vec{\mathbf{0}}:=(0,0,\dots,0)^{\mathrm{T}}$ $(\in\mathbb{R}^N)$. *Set*

$$
\begin{aligned}
(\widehat{Q}{}_0^e)_{11} &:= (\overset{e}{m}{}^{0}_{0})_{11}\Big(1+(\mathcal{R}_0^{e,0}(n))_{11}\Big)+(\mathcal{R}_0^{e,0}(n))_{12}(\overset{e}{m}{}^{0}_{0})_{21},\\
(\widehat{Q}{}_0^e)_{12} &:= (\overset{e}{m}{}^{0}_{0})_{12}\Big(1+(\mathcal{R}_0^{e,0}(n))_{11}\Big)+(\mathcal{R}_0^{e,0}(n))_{12}(\overset{e}{m}{}^{0}_{0})_{22},\\
(\widehat{Q}{}_0^e)_{21} &:= (\overset{e}{m}{}^{0}_{0})_{21}\Big(1+(\mathcal{R}_0^{e,0}(n))_{22}\Big)+(\mathcal{R}_0^{e,0}(n))_{21}(\overset{e}{m}{}^{0}_{0})_{11},\\
(\widehat{Q}{}_0^e)_{22} &:= (\overset{e}{m}{}^{0}_{0})_{22}\Big(1+(\mathcal{R}_0^{e,0}(n))_{22}\Big)+(\mathcal{R}_0^{e,0}(n))_{21}(\overset{e}{m}{}^{0}_{0})_{12},\\
(\widehat{Q}{}_1^e)_{11} &:= (\overset{e}{m}{}^{0}_{1})_{11}\Big(1+(\mathcal{R}_0^{e,0}(n))_{11}\Big)+(\mathcal{R}_0^{e,0}(n))_{12}(\overset{e}{m}{}^{0}_{1})_{21}+(\mathcal{R}_1^{e,0}(n))_{11}(\overset{e}{m}{}^{0}_{0})_{11}\\
&\quad + (\mathcal{R}_1^{e,0}(n))_{12}(\overset{e}{m}{}^{0}_{0})_{21},\\
(\widehat{Q}{}_1^e)_{12} &:= (\overset{e}{m}{}^{0}_{1})_{12}\Big(1+(\mathcal{R}_0^{e,0}(n))_{11}\Big)+(\mathcal{R}_0^{e,0}(n))_{12}(\overset{e}{m}{}^{0}_{1})_{22}+(\mathcal{R}_1^{e,0}(n))_{11}(\overset{e}{m}{}^{0}_{0})_{12}\\
&\quad + (\mathcal{R}_1^{e,0}(n))_{12}(\overset{e}{m}{}^{0}_{0})_{22},\\
(\widehat{Q}{}_1^e)_{21} &:= (\overset{e}{m}{}^{0}_{1})_{21}\Big(1+(\mathcal{R}_0^{e,0}(n))_{22}\Big)+(\mathcal{R}_0^{e,0}(n))_{21}(\overset{e}{m}{}^{0}_{1})_{11}+(\mathcal{R}_1^{e,0}(n))_{21}(\overset{e}{m}{}^{0}_{0})_{11}\\
&\quad + (\mathcal{R}_1^{e,0}(n))_{22}(\overset{e}{m}{}^{0}_{0})_{21},\\
(\widehat{Q}{}_1^e)_{22} &:= (\overset{e}{m}{}^{0}_{1})_{22}\Big(1+(\mathcal{R}_0^{e,0}(n))_{22}\Big)+(\mathcal{R}_0^{e,0}(n))_{21}(\overset{e}{m}{}^{0}_{1})_{12}+(\mathcal{R}_1^{e,0}(n))_{21}(\overset{e}{m}{}^{0}_{0})_{12}\\
&\quad + (\mathcal{R}_1^{e,0}(n))_{22}(\overset{e}{m}{}^{0}_{0})_{22},\\
(\widehat{Q}{}_2^e)_{11} &:= (\overset{e}{m}{}^{0}_{2})_{11}\Big(1+(\mathcal{R}_0^{e,0}(n))_{11}\Big)+(\mathcal{R}_0^{e,0}(n))_{12}(\overset{e}{m}{}^{0}_{2})_{21}+(\mathcal{R}_1^{e,0}(n))_{11}(\overset{e}{m}{}^{0}_{1})_{11}\\
&\quad + (\mathcal{R}_1^{e,0}(n))_{12}(\overset{e}{m}{}^{0}_{1})_{21}+(\mathcal{R}_2^{e,0}(n))_{11}(\overset{e}{m}{}^{0}_{0})_{11}+(\mathcal{R}_2^{e,0}(n))_{12}(\overset{e}{m}{}^{0}_{0})_{21},\\
(\widehat{Q}{}_2^e)_{12} &:= (\overset{e}{m}{}^{0}_{2})_{12}\Big(1+(\mathcal{R}_0^{e,0}(n))_{11}\Big)+(\mathcal{R}_0^{e,0}(n))_{12}(\overset{e}{m}{}^{0}_{2})_{22}+(\mathcal{R}_1^{e,0}(n))_{11}(\overset{e}{m}{}^{0}_{1})_{12}
\end{aligned}
$$

$$
\begin{aligned}
&+(\mathcal{R}_1^{e,0}(n))_{12}(\overset{e}{m}{}_1^0)_{22}+(\mathcal{R}_2^{e,0}(n))_{11}(\overset{e}{m}{}_0^0)_{12}+(\mathcal{R}_2^{e,0}(n))_{12}(\overset{e}{m}{}_0^0)_{22},\\
(\widehat{Q}_2^e)_{21} :=&\, (\overset{e}{m}{}_2^0)_{21}\Big(1+(\mathcal{R}_0^{e,0}(n))_{22}\Big)+(\mathcal{R}_0^{e,0}(n))_{21}(\overset{e}{m}{}_2^0)_{11}+(\mathcal{R}_1^{e,0}(n))_{21}(\overset{e}{m}{}_1^0)_{11}\\
&+(\mathcal{R}_1^{e,0}(n))_{22}(\overset{e}{m}{}_1^0)_{21}+(\mathcal{R}_2^{e,0}(n))_{21}(\overset{e}{m}{}_0^0)_{11}+(\mathcal{R}_2^{e,0}(n))_{22}(\overset{e}{m}{}_0^0)_{21},\\
(\widehat{Q}_2^e)_{22} :=&\, (\overset{e}{m}{}_2^0)_{22}\Big(1+(\mathcal{R}_0^{e,0}(n))_{22}\Big)+(\mathcal{R}_0^{e,0}(n))_{21}(\overset{e}{m}{}_2^0)_{12}+(\mathcal{R}_1^{e,0}(n))_{21}(\overset{e}{m}{}_1^0)_{12}\\
&+(\mathcal{R}_1^{e,0}(n))_{22}(\overset{e}{m}{}_1^0)_{22}+(\mathcal{R}_2^{e,0}(n))_{21}(\overset{e}{m}{}_0^0)_{12}+(\mathcal{R}_2^{e,0}(n))_{22}(\overset{e}{m}{}_0^0)_{22}.
\end{aligned}
$$

Let $\overset{e}{\mathrm{Y}}\colon \mathbb{C}\setminus\mathbb{R}\to \mathrm{SL}_2(\mathbb{C})$ *be the solution of* **RHP1**. *Then,*

$$\overset{e}{\mathrm{Y}}(z)z^{n\sigma_3} \underset{z\to 0}{=} \mathrm{Y}_0^{e,0}+z\mathrm{Y}_1^{e,0}+z^2\mathrm{Y}_2^{e,0}+\mathcal{O}(z^3),$$

where

$$
\begin{aligned}
(\mathrm{Y}_0^{e,0})_{11} &= (\widehat{Q}_0^e)_{11}\mathrm{e}^{2n(\int_{J_e}\ln(|s|)\psi_V^e(s)\,\mathrm{d}s+\mathrm{i}\pi\int_{J_e\cap\mathbb{R}_+}\psi_V^e(s)\,\mathrm{d}s)},\\
(\mathrm{Y}_0^{e,0})_{12} &= (\widehat{Q}_0^e)_{12}\mathrm{e}^{n(\ell_e-2\int_{J_e}\ln(|s|)\psi_V^e(s)\,\mathrm{d}s-2\pi\mathrm{i}\int_{J_e\cap\mathbb{R}_+}\psi_V^e(s)\,\mathrm{d}s)},\\
(\mathrm{Y}_0^{e,0})_{21} &= (\widehat{Q}_0^e)_{21}\mathrm{e}^{-n(\ell_e-2\int_{J_e}\ln(|s|)\psi_V^e(s)\,\mathrm{d}s-2\pi\mathrm{i}\int_{J_e\cap\mathbb{R}_+}\psi_V^e(s)\,\mathrm{d}s)},\\
(\mathrm{Y}_0^{e,0})_{22} &= (\widehat{Q}_0^e)_{22}\mathrm{e}^{-2n(\int_{J_e}\ln(|s|)\psi_V^e(s)\,\mathrm{d}s+\mathrm{i}\pi\int_{J_e\cap\mathbb{R}_+}\psi_V^e(s)\,\mathrm{d}s)},\\
(\mathrm{Y}_1^{e,0})_{11} &= \left((\widehat{Q}_1^e)_{11}-2n(\widehat{Q}_0^e)_{11}\int_{J_e}s^{-1}\psi_V^e(s)\,\mathrm{d}s\right)\mathrm{e}^{2n(\int_{J_e}\ln(|s|)\psi_V^e(s)\,\mathrm{d}s+\mathrm{i}\pi\int_{J_e\cap\mathbb{R}_+}\psi_V^e(s)\,\mathrm{d}s)},\\
(\mathrm{Y}_1^{e,0})_{12} &= \left((\widehat{Q}_1^e)_{12}+2n(\widehat{Q}_0^e)_{12}\int_{J_e}s^{-1}\psi_V^e(s)\,\mathrm{d}s\right)\mathrm{e}^{n(\ell_e-2\int_{J_e}\ln(|s|)\psi_V^e(s)\,\mathrm{d}s-2\pi\mathrm{i}\int_{J_e\cap\mathbb{R}_+}\psi_V^e(s)\,\mathrm{d}s)},\\
(\mathrm{Y}_1^{e,0})_{21} &= \left((\widehat{Q}_1^e)_{21}-2n(\widehat{Q}_0^e)_{21}\int_{J_e}s^{-1}\psi_V^e(s)\,\mathrm{d}s\right)\mathrm{e}^{-n(\ell_e-2\int_{J_e}\ln(|s|)\psi_V^e(s)\,\mathrm{d}s-2\pi\mathrm{i}\int_{J_e\cap\mathbb{R}_+}\psi_V^e(s)\,\mathrm{d}s)},\\
(\mathrm{Y}_1^{e,0})_{22} &= \left((\widehat{Q}_1^e)_{22}+2n(\widehat{Q}_0^e)_{22}\int_{J_e}s^{-1}\psi_V^e(s)\,\mathrm{d}s\right)\mathrm{e}^{-2n(\int_{J_e}\ln(|s|)\psi_V^e(s)\,\mathrm{d}s+\mathrm{i}\pi\int_{J_e\cap\mathbb{R}_+}\psi_V^e(s)\,\mathrm{d}s)},\\
(\mathrm{Y}_2^{e,0})_{11} &= \Bigg((\widehat{Q}_2^e)_{11}-2n(\widehat{Q}_1^e)_{11}\int_{J_e}s^{-1}\psi_V^e(s)\,\mathrm{d}s+(\widehat{Q}_0^e)_{11}\bigg(2n^2\left(\int_{J_e}s^{-1}\psi_V^e(s)\,\mathrm{d}s\right)^2\\
&\quad-n\int_{J_e}s^{-2}\psi_V^e(s)\,\mathrm{d}s\bigg)\Bigg)\mathrm{e}^{2n(\int_{J_e}\ln(|s|)\psi_V^e(s)\,\mathrm{d}s+\mathrm{i}\pi\int_{J_e\cap\mathbb{R}_+}\psi_V^e(s)\,\mathrm{d}s)},\\
(\mathrm{Y}_2^{e,0})_{12} &= \Bigg((\widehat{Q}_2^e)_{12}+2n(\widehat{Q}_1^e)_{12}\int_{J_e}s^{-1}\psi_V^e(s)\,\mathrm{d}s+(\widehat{Q}_0^e)_{12}\bigg(2n^2\left(\int_{J_e}s^{-1}\psi_V^e(s)\,\mathrm{d}s\right)^2\\
&\quad+n\int_{J_e}s^{-2}\psi_V^e(s)\,\mathrm{d}s\bigg)\Bigg)\mathrm{e}^{n(\ell_e-2\int_{J_e}\ln(|s|)\psi_V^e(s)\,\mathrm{d}s-2\pi\mathrm{i}\int_{J_e\cap\mathbb{R}_+}\psi_V^e(s)\,\mathrm{d}s)},\\
(\mathrm{Y}_2^{e,0})_{21} &= \Bigg((\widehat{Q}_2^e)_{21}-2n(\widehat{Q}_1^e)_{21}\int_{J_e}s^{-1}\psi_V^e(s)\,\mathrm{d}s+(\widehat{Q}_0^e)_{21}\bigg(2n^2\left(\int_{J_e}s^{-1}\psi_V^e(s)\,\mathrm{d}s\right)^2\\
&\quad-n\int_{J_e}s^{-2}\psi_V^e(s)\,\mathrm{d}s\bigg)\Bigg)\mathrm{e}^{-n(\ell_e-2\int_{J_e}\ln(|s|)\psi_V^e(s)\,\mathrm{d}s-2\pi\mathrm{i}\int_{J_e\cap\mathbb{R}_+}\psi_V^e(s)\,\mathrm{d}s)},\\
(\mathrm{Y}_2^{e,0})_{22} &= \Bigg((\widehat{Q}_2^e)_{22}+2n(\widehat{Q}_1^e)_{22}\int_{J_e}s^{-1}\psi_V^e(s)\,\mathrm{d}s+(\widehat{Q}_0^e)_{22}\bigg(2n^2\left(\int_{J_e}s^{-1}\psi_V^e(s)\,\mathrm{d}s\right)^2\\
&\quad+n\int_{J_e}s^{-2}\psi_V^e(s)\,\mathrm{d}s\bigg)\Bigg)\mathrm{e}^{-2n(\int_{J_e}\ln(|s|)\psi_V^e(s)\,\mathrm{d}s+\mathrm{i}\pi\int_{J_e\cap\mathbb{R}_+}\psi_V^e(s)\,\mathrm{d}s)}.
\end{aligned}
$$

Proof. Let $\mathcal{R}^e\colon \mathbb{C}\setminus\widetilde{\Sigma}_p^e\to\mathrm{SL}_2(\mathbb{C})$ be the solution of the RHP $(\mathcal{R}^e(z),v_{\mathcal{R}}^e(z),\widetilde{\Sigma}_p^e)$ formulated in Proposition 5.2 with $n\to\infty$ asymptotics given in Lemma 5.3. For $|z|\ll\min_{j=1,\dots,N+1}\{|b_{j-1}^e-a_j^e|\}$, via the expansion $\frac{1}{z-s}=-\sum_{k=0}^{l}\frac{z^k}{s^{k+1}}+\frac{z^{l+1}}{s^{l+1}(z-s)}$, $l\in\mathbb{Z}_0^+$, where $s\in\{b_{j-1}^e,a_j^e\}$, $j=1,\dots,N+1$, one obtains the asymptotics for $\mathcal{R}^e(z)$ stated in the Proposition.

Let $\overset{e}{m}{}^{\infty} : \mathbb{C} \setminus J_e^{\infty} \to \mathrm{SL}_2(\mathbb{C})$ solve the RHP $(\overset{e}{m}{}^{\infty}(z), J_e^{\infty}, \overset{e}{v}{}^{\infty}(z))$ formulated in Lemma 4.3 with (unique) solution given by Lemma 4.5. In order to obtain small-z asymptotics of $\overset{e}{m}{}^{\infty}(z)$, one needs small-$z$ asymptotics of $(\gamma^e(z))^{\pm 1}$ and $\frac{\boldsymbol{\theta}^e(\varepsilon_1 \boldsymbol{u}^e(z) - \frac{n}{2\pi}\boldsymbol{\Omega}^e + \varepsilon_2 \boldsymbol{d}_e)}{\boldsymbol{\theta}^e(\varepsilon_1 \boldsymbol{u}^e(z) + \varepsilon_2 \boldsymbol{d}_e)}$, $\varepsilon_1, \varepsilon_2 = \pm 1$. Consider, say, and without loss of generality, $z \to 0$ asymptotics for $z \in \mathbb{C}_+$ (designated $z \to 0^+$), where, by definition, $\sqrt{\star(z)} := +\sqrt{\star(z)}$: equivalently, one may consider $z \to 0$ asymptotics for $z \in \mathbb{C}_-$ (designated $z \to 0^-$); however, recalling that $\sqrt{\star(z)} \restriction_{\mathbb{C}_+} = -\sqrt{\star(z)} \restriction_{\mathbb{C}_-}$, one obtains (in either case, and via the sheet-interchange index) the same $z \to 0$ asymptotics (for $\overset{e}{m}{}^{\infty}(z)$). Recall the expression for $\gamma^e(z)$ given in Lemma 4.4: for $|z| \ll \min_{j=1,\dots,N+1}\{|b_{j-1}^e - a_j^e|\}$, via the expansions $\frac{1}{z-s} = -\sum_{k=0}^{l} \frac{z^k}{s^{k+1}} + \frac{z^{l+1}}{s^{l+1}(z-s)}$, $l \in \mathbb{Z}_0^+$, and $\ln(s-z) =_{|z|\to 0} \ln(s) - \sum_{k=1}^{\infty} \frac{1}{k}(\frac{z}{s})^k$, where $s \in \{b_{j-1}^e, a_j^e\}$, $j = 1, \dots, N+1$, one shows that, upon defining $\gamma^e(0)$ as in the Proposition,

$$(\gamma^e(z))^{\pm 1} \underset{z\to 0^+}{=} (\gamma_0^e)^{\pm 1}\left(1 + z\left(\pm \frac{1}{4}\sum_{k=1}^{N+1}\left(\frac{1}{a_k^e} - \frac{1}{b_{k-1}^e}\right)\right) + z^2\left(\pm \frac{1}{8}\sum_{k=1}^{N+1}\left(\frac{1}{(a_k^e)^2} - \frac{1}{(b_{k-1}^e)^2}\right)\right.\right.$$
$$\left.\left. + \frac{1}{32}\left(\sum_{k=1}^{N+1}\left(\frac{1}{a_k^e} - \frac{1}{b_{k-1}^e}\right)\right)^2\right) + \mathcal{O}(z^3)\right),$$

whence

$$\frac{1}{2}(\gamma^e(z) + (\gamma^e(z))^{-1}) \underset{z\to 0^+}{=} \frac{(\gamma_0^e + (\gamma_0^e)^{-1})}{2} + z\left(\left(\frac{\gamma_0^e - (\gamma_0^e)^{-1}}{8}\right)\sum_{k=1}^{N+1}\left(\frac{1}{a_k^e} - \frac{1}{b_{k-1}^e}\right)\right)$$
$$+ z^2\left(\left(\frac{\gamma_0^e - (\gamma_0^e)^{-1}}{16}\right)\sum_{k=1}^{N+1}\left(\frac{1}{(a_k^e)^2} - \frac{1}{(b_{k-1}^e)^2}\right) + \left(\frac{\gamma_0^e + (\gamma_0^e)^{-1}}{64}\right)\right.$$
$$\left. \times \left(\sum_{k=1}^{N+1}\left(\frac{1}{a_k^e} - \frac{1}{b_{k-1}^e}\right)\right)^2\right) + \mathcal{O}(z^3),$$

and

$$\frac{1}{2\mathrm{i}}(\gamma^e(z) - (\gamma^e(z))^{-1}) \underset{z\to 0^+}{=} \frac{(\gamma_0^e - (\gamma_0^e)^{-1})}{2\mathrm{i}} + z\left(\left(\frac{\gamma_0^e + (\gamma_0^e)^{-1}}{8\mathrm{i}}\right)\sum_{k=1}^{N+1}\left(\frac{1}{a_k^e} - \frac{1}{b_{k-1}^e}\right)\right)$$
$$+ z^2\left(\left(\frac{\gamma_0^e + (\gamma_0^e)^{-1}}{16\mathrm{i}}\right)\sum_{k=1}^{N+1}\left(\frac{1}{(a_k^e)^2} - \frac{1}{(b_{k-1}^e)^2}\right) + \left(\frac{\gamma_0^e - (\gamma_0^e)^{-1}}{64\mathrm{i}}\right)\right.$$
$$\left. \times \left(\sum_{k=1}^{N+1}\left(\frac{1}{a_k^e} - \frac{1}{b_{k-1}^e}\right)\right)^2\right) + \mathcal{O}(z^3).$$

Recall from Lemma 4.5 that $\boldsymbol{u}^e(z) := \int_{a_{N+1}^e}^{z} \boldsymbol{\omega}^e$ ($\in \mathrm{Jac}(\mathcal{Y}_e)$, with $\mathcal{Y}_e := \{(y,z);\ y^2 = R_e(z)\}$), where $\boldsymbol{\omega}^e$, the associated normalised basis of holomorphic one-forms of $\mathcal{Y}_e$, is given by $\boldsymbol{\omega}^e = (\omega_1^e, \omega_2^e, \dots, \omega_N^e)$, with $\omega_j^e := \sum_{k=1}^{N} c_{jk}^e (\prod_{i=1}^{N+1}(z - b_{i-1}^e)(z - a_i^e))^{-1/2} z^{N-k}\,\mathrm{d}z$, $j = 1, \dots, N$, where c_{jk}^e, $j,k = 1, \dots, N$, are obtained from Equations (E1) and (E2). Writing

$$\boldsymbol{u}^e(z) = \left(\int_{a_{N+1}^e}^{0^+} + \int_{0^+}^{z}\right)\boldsymbol{\omega}^e = \boldsymbol{u}_+^e(0) + \int_{0^+}^{z}\boldsymbol{\omega}^e,$$

where $\boldsymbol{u}_+^e(0) := \int_{a_{N+1}^e}^{0^+} \boldsymbol{\omega}^e$, for $|z| \ll \min_{j=1,\dots,N+1}\{|b_{j-1}^e - a_j^e|\}$, via the expansions $\frac{1}{z-s} = -\sum_{k=0}^{l} \frac{z^k}{s^{k+1}} + \frac{z^{l+1}}{s^{l+1}(z-s)}$, $l \in \mathbb{Z}_0^+$, and $\ln(s-z) =_{|z|\to 0} \ln(s) - \sum_{k=1}^{\infty}\frac{1}{k}(\frac{z}{s})^k$, where $s \in \{b_{k-1}^e, a_k^e\}$, $k = 1, \dots, N+1$, one shows that, for $j = 1, \dots, N$,

$$\omega_j^e \underset{z\to 0^+}{=} (-1)^{\mathcal{N}_+}\left(\prod_{i=1}^{N+1}|b_{i-1}^e a_i^e|\right)^{-1/2}\left(c_{jN}^e\,\mathrm{d}z + \left(c_{jN-1}^e + \frac{c_{jN}^e}{2}\sum_{k=1}^{N+1}\left(\frac{1}{b_{k-1}^e} + \frac{1}{a_k^e}\right)\right)z\,\mathrm{d}z + \mathcal{O}(z^2\,\mathrm{d}z)\right),$$

where $\mathcal{N}_{+} \in \{0,\dots,N+1\}$ is the number of bands to the right of $z=0$, whence

$$\int_{0^{+}}^{z} \omega_{j}^{e} \underset{z\to 0^{+}}{=} (-1)^{\mathcal{N}_{+}} \left(\prod_{i=1}^{N+1} |b_{i-1}^{e} a_{i}^{e}| \right)^{-1/2} \left(c_{jN}^{e} z + \frac{1}{2} \left(c_{jN-1}^{e} + \frac{c_{jN}^{e}}{2} \sum_{k=1}^{N+1} \left(\frac{1}{b_{k-1}^{e}} + \frac{1}{a_{k}^{e}} \right) \right) z^{2} + O(z^{3}) \right)$$
$$=: \widehat{\alpha}_{0,j}^{e} z + \widehat{\beta}_{0,j}^{e} z^{2} + O(z^{3}).$$

Defining $\theta_{0}^{e}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{e})$, $\widetilde{\alpha}_{0}^{e}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{e})$, and $\beta_{0}^{e}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{e})$, $\varepsilon_1,\varepsilon_2=\pm 1$, as in the Proposition, recalling that $\boldsymbol{\omega}^{e}=(\omega_1^{e},\omega_2^{e},\dots,\omega_N^{e})$, and that the associated $N\times N$ Riemann matrix of $\boldsymbol{\beta}^{e}$-periods, that is, $\tau^{e}=(\tau^{e})_{i,j=1,\dots,N}:=(\oint_{\boldsymbol{\beta}_{j}^{e}} \omega_{i}^{e})_{i,j=1,\dots,N}$, is non-degenerate, symmetric, and $-\mathrm{i}\tau^{e}$ is positive definite, via the above asymptotic (as $z\to 0^{+}$) expansion for $\int_{0^{+}}^{z}\omega_{j}^{e}$, $j=1,\dots,N$, one shows that

$$\frac{\boldsymbol{\theta}^{e}(\varepsilon_1 \boldsymbol{u}^{e}(z) - \frac{n}{2\pi}\boldsymbol{\Omega}^{e} + \varepsilon_2 \boldsymbol{d}_{e})}{\boldsymbol{\theta}^{e}(\varepsilon_1 \boldsymbol{u}^{e}(z) + \varepsilon_2 \boldsymbol{d}_{e})} \underset{z\to 0^{+}}{=} F_0^{e} + zF_1^{e} + z^{2}F_2^{e} + O(z^{3}),$$

where

$$F_0^{e} := \frac{\theta_0^{e}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{e})}{\theta_0^{e}(\varepsilon_1,\varepsilon_2,\vec{\boldsymbol{0}})},$$
$$F_1^{e} := \frac{\widetilde{\alpha}_0^{e}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{e})\theta_0^{e}(\varepsilon_1,\varepsilon_2,\vec{\boldsymbol{0}}) - \theta_0^{e}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{e})\widetilde{\alpha}_0^{e}(\varepsilon_1,\varepsilon_2,\vec{\boldsymbol{0}})}{(\theta_0^{e}(\varepsilon_1,\varepsilon_2,\vec{\boldsymbol{0}}))^{2}},$$
$$F_2^{e} := \Big(\theta_0^{e}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{e}) \Big((\widetilde{\alpha}_0^{e}(\varepsilon_1,\varepsilon_2,\vec{\boldsymbol{0}}))^{2} - \beta_0^{e}(\varepsilon_1,\varepsilon_2,\vec{\boldsymbol{0}})\theta_0^{e}(\varepsilon_1,\varepsilon_2,\vec{\boldsymbol{0}}) \Big) - \widetilde{\alpha}_0^{e}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{e})$$
$$\times \widetilde{\alpha}_0^{e}(\varepsilon_1,\varepsilon_2,\vec{\boldsymbol{0}})\theta_0^{e}(\varepsilon_1,\varepsilon_2,\vec{\boldsymbol{0}}) + \beta_0^{e}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{e})(\theta_0^{e}(\varepsilon_1,\varepsilon_2,\vec{\boldsymbol{0}}))^{2} \Big) (\theta_0^{e}(\varepsilon_1,\varepsilon_2,\vec{\boldsymbol{0}}))^{-3},$$

with $\vec{\boldsymbol{0}} := (0,0,\dots,0)^{\mathrm{T}}$ ($\in \mathbb{R}^{N}$). Via the above asymptotic (as $z\to 0^{+}$) expansions for $\frac{1}{2}(\gamma^{e}(z)+(\gamma^{e}(z))^{-1})$, $\frac{1}{2\mathrm{i}}(\gamma^{e}(z)-(\gamma^{e}(z))^{-1})$, and $\frac{\boldsymbol{\theta}^{e}(\varepsilon_1 \boldsymbol{u}^{e}(z) - \frac{n}{2\pi}\boldsymbol{\Omega}^{e} + \varepsilon_2 \boldsymbol{d}_{e})}{\boldsymbol{\theta}^{e}(\varepsilon_1 \boldsymbol{u}^{e}(z) + \varepsilon_2 \boldsymbol{d}_{e})}$, one arrives at, upon recalling the expression for $\overset{e}{m}{}^{\infty}(z)$ given in Lemma 4.5, the asymptotic expansion for $\overset{e}{m}{}^{\infty}(z)$ stated in the Proposition.

Let $\overset{e}{\mathrm{Y}}\colon \mathbb{C}\setminus\mathbb{R}\to \mathrm{SL}_2(\mathbb{C})$ be the (unique) solution of **RHP1**, that is, $(\overset{e}{\mathrm{Y}}(z), \mathrm{I}+\exp(-n\widetilde{V}(z))\sigma_{+}, \mathbb{R})$. Recall, also, that, for $z\in \Upsilon_1^{e}\cup\Upsilon_2^{e}$ (Figure 7),

$$\overset{e}{\mathrm{Y}}(z) = \mathrm{e}^{\frac{n\ell_e}{2}\mathrm{ad}(\sigma_3)} \mathcal{R}^{e}(z) \overset{e}{m}{}^{\infty}(z) \mathrm{e}^{n(g^{e}(z)+\int_{J_e}\ln(s)\psi_V^{e}(s)\,\mathrm{d}s)\sigma_3}:$$

consider, say, and without loss of generality, small-z asymptotics for $\overset{e}{\mathrm{Y}}(z)$ for $z\in\Upsilon_1^{e}$. Recalling from Lemma 3.4 that $g^{e}(z) := \int_{J_e} \ln((z-s)^{2}(zs)^{-1})\psi_V^{e}(s)\,\mathrm{d}s$, $z\in\mathbb{C}\setminus(-\infty,\max\{0,a_{N+1}^{e}\})$, for $|z|\ll \min_{j=1,\dots,N+1}\{|b_{j-1}^{e}-a_{j}^{e}|\}$, in particular, $|z/s|\ll 1$ with $s\in J_e$, and noting that $\int_{J_e}\psi_V^{e}(s)\,\mathrm{d}s=1$ and $\int_{J_e} s^{-m}\psi_V^{e}(s)\,\mathrm{d}s<\infty$, $m\in\mathbb{N}$, via the expansions $\frac{1}{z-s} = -\sum_{k=0}^{l}\frac{z^{k}}{s^{k+1}} + \frac{z^{l+1}}{s^{l+1}(z-s)}$, $l\in\mathbb{Z}_0^{+}$, and $\ln(s-z) =_{|z|\to 0} \ln(s) - \sum_{k=1}^{\infty}\frac{1}{k}(\frac{z}{s})^{k}$, one shows that

$$g^{e}(z) \underset{\mathbb{C}_{\pm}\ni z\to 0}{=} -\ln(z) - Q_e + 2\int_{J_e}\ln(|s|)\psi_V^{e}(s)\,\mathrm{d}s \pm 2\pi\mathrm{i}\int_{J_e\cap\mathbb{R}_{+}}\psi_V^{e}(s)\,\mathrm{d}s$$
$$+ z\left(-2\int_{J_e} s^{-1}\psi_V^{e}(s)\,\mathrm{d}s\right) + z^{2}\left(-\int_{J_e} s^{-2}\psi_V^{e}(s)\,\mathrm{d}s\right) + O(z^{3}),$$

where $\int_{J_e\cap\mathbb{R}_{+}}\psi_V^{e}(s)\,\mathrm{d}s$ is given in the proof of Lemma 3.4. (Explicit expressions for $\int_{J_e} s^{-k}\psi_V^{e}(s)\,\mathrm{d}s$, $k=1,2$, are given in Remark 3.2.) Using the asymptotic (as $z\to 0$) expansions for $g^{e}(z)$, $\mathcal{R}^{e}(z)$, and $\overset{e}{m}{}^{\infty}(z)$ derived above, upon recalling the formula for $\overset{e}{\mathrm{Y}}(z)$, one arrives at, after a matrix-multiplication argument, the asymptotic expansion for $\overset{e}{\mathrm{Y}}(z)z^{n\sigma_3}$ stated in the Proposition. $\square$

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